75 Marks → Below

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Engineering Maths + Discrete Maths → 21-22 Marks

+ Numerical Ability

↓

35 marks Total

1.) Set Theory → 40%.

2.) Graph Theory → 20%.

3.) Combinotorics → 20%.

4.) Mathematical logic → 20%.

Engg. Maths

1) Linear Algebra

2) Probability

3) Calculus

Kennath Rossen

↓

Discrete Mathematic & Application

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Concept — ①

Worked out example

1987 - 2021

Lecture - 1

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11/4/21

Discrete Mathematics

Set Theory

1.) Sets (12 Q)

- Basics — 2
- Powerset — 5
- Venn Diagram — 4
- Multi Set — 1

2.) Relations (20 Q).

- Reflexive
- Irreflexive
- Symmetric
- Antisymmetric
- Asymmetric
- Transitive
- Equivalence

5) Functions (14 Q)

3.) Lattice (5 Q)

- Partial Order
- POSET
- Totally Order
- TOSET
- Lattice
- Types of lattice

4) Groups (16 Q)

- Definitions
- Properties
- Subgroup
- Lagrange's Theorem *****
- Cyclic group

Set :- well defined ~~colll~~ collection of unordered distinct object.

Ex :- The collection of all tall boys in the class is not a set

[we don't know that which height we can treat as tall, so it is not well defined].

Ex :- The collection of all tall boys in the class whose height ≥ 165cm is a set

Ex :- The collection of all letters of the word 'flow'.

{f, l, o, w} = {o, l, w, f} = {l, o, w, f}

~~Ex~~ :- which of the following sets are equal?

1) The collection of all letters of the word 'follow'

$A = \{f, o, l, w\}$

2) The collection all letters of 'woolf'

$B = \{w, o, l, f\}$

3) The collection of all letters of 'flow'

$C = \{f, l, o, w\}$

$\therefore A = B = C$

Empty set :-

It is denoted by ϕ or $\{\}$

- $|\phi| = 0$ → cardinality
  no. of elements present.
- $|\{\phi\}| = 1$

$A = \{1, 2, 3, 4\}$ = set of elements

$|A| = 4$

Cardinality of B is 3

$B = \{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$

$|B| = 3$ = set of sets

$C = \{(1, 2), (3, 4)\}$

$|C| = 2$ = set of ordered pair

$D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in \mathbb{R} \right\}$

$|D| = \infty$ → as infinite no. of elements are present in real no.

= set of ~~mat~~ 2×2 matrices

i) A = Set of all negative numbers which are greater than 5

= { }

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Subset

let A, B are two sets

If every element of 'A' is also an element of B then 'A' is called subset of B, it is denoted by

$A \subseteq B$ (Read it A is ~~denoted~~ subset of B)

Ex:- $A = \{1, 2, 3, 4\}$

$B = \{-2, -1, 0, 1, 2, 3, 4, 5\}$

$\therefore A \subseteq B$

Note :-

i) $\phi \subseteq A$ [A → every set (Any)]

ii) $A \subseteq A$

Ex :-

$A = \{1, 2, 3, 4\}$

$B = \{x \mid x \in \mathbb{N} \text{ and } x \leq 5\}$

$= \{1, 2, 3, 4, 5\}$

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A is proper subset of B

$A \subset B$ (when we know B has some extra element)

here we can write $A \subseteq B$ also nothing is wrong if we write like that.

$C = \{x / x \in N \text{ and } x \leq 4\}$

$= \{1, 2, 3, 4\}$

$A \subseteq C$ means

either $A \subset C$ (or) $A = C$

Power set :-

The collection of all possible subsets of a given set is called powerset of the given set.

$A = \{1, 2\}$

$P(A) = \{\overset{{}^2C_0}{\phi}, \overset{{}^2C_1}{\overbrace{\{1\}, \{2\}}}, \overset{{}^2C_2}{\{1, 2\}}\}$

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$A = \{1, 2, 3\}$

$^3C_0$ $^3C_1$ $^3C_2$ $^3C_3$ $\{1,3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

$|P(A)| = {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 8$

In general, if $|A| = n$, then

$|P(A)| = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$

$= 2^n$ (∵ Using Binomial Theorem)

Basics

0/1 x_1	0/1 x_2
0	0
0	1
1	0
1	1

$= 4 = \overset{\text{no. of elements}}{2^{2}}$

$\underset{\text{either 0 or 1 (2 possibilities)}}{2}$

x_1	x_2	x_3
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

$= 8 = 2^3$

0 → absent
1 → present

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0/1	0/1		0/1
x_1	x_2	- - -	x_n

$= 2^n$

n → no. of elements

2 → either absent (0) or present (1)

If $|A| = n$, then every element of the set 'A' will appear in two ways in P(A) i.e either that particular element is present (or) absent in the subset.

∴ $|P(A)| = 2^n$

Gate 2015

Q.1) The cardinality of the power set of $\{0, 1, 2, 3, \dots, 10\}$ is ______

Powerset contain $= 2^{11}$ elements

(∵ If $|A| = n$, then $|P(A)| = 2^n$).

Q.2 Let $A = \{\phi, 2, \{3,4\}, \{4\}, \{5,6,7\}\}$

Which of the following is/are True?

i) $\{2, \{4\}\} \subseteq A$ iii) $\{\{3,4\}\} \subseteq A$

ii) $\{\phi, \{2\}, \{4\}\} \subseteq A$ iv) $\{\{3,4\}, \{4\}\} \subseteq A$

v) $\{\{3,4\},4\} \subseteq A$ x) $\{5,6,7\} \in A$

vi) $\{2,\{4\}\} \in A$ xi) $\{\{5,6,7\}\} \in A$

vii) $\{\{3\},\{4\}\} \in A$ xii) $\{\{5,6,7\}\} \subset A$

viii) $\{\{3,4\}\} \in A$

iv) $\{3,4\} \in A$

Ans (i), [illegible], (iii), (iv), (ix), (x), (xii) → TRUE

$\{3,4\} \in A$

↳ same element should be present in the set

$\subseteq$ → Remove outer braces, verify each element is present or not

G:2015

Q. For a Set 'A', the powerset of A is denoted by 2^A.
If $A = \{5,\{6\},\{7\}\}$.
Which of the following options are TRUE

i) $\phi \in 2^A$ (ii) $\phi \subseteq 2^A$

iii) $\{5,\{6\}\} \in 2^A$ (iv) $\{5,\{6\}\} \subseteq 2^A$

A) I and II only
B) II and III only
C) I, II and III only
D) I, II, and IV only

→ C option is correct

G: 2000

Q Let P(S) denotes the powerset of set S. which of the following is always TRUE.

a) $P(P(S)) = P(S)$

b) $P(S) \cap P(P(S)) = \{\phi\}$

c) $P(S) \cap S = P(S)$

d) $S \notin P(S)$

$|S| = n, \; |P(S)| = 2^n, \; |P(P(S))| = 2^{2^n}$

a) → ✗ $2^n \neq 2^{2^n}$

(b) $A = B$ {$|A| = |B|$ and A, B are having same elements}

c) → ✗ as intersection is there so atmost elements are n

d) → ✗ as $S \in P(S)$

$S = \{a, b\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

$$P(P(S)) = \{\phi, \{\phi\}, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\},$$
$$\{\phi, \{a\}\}, \{\{a\},\{b\}\}, \{\phi, \{b\}\}$$
$$\{\phi, \{a,b\}\}, \{\{a\}, \{a,b\}\}$$
$$\{\{b\}, \{a,b\}\}, \{\phi, \{a\}, \{b\}\},$$
$$\{\phi, \{a\}, \{a,b\}\}, \{\phi, \{b\}, \{a,b\}\}$$
$$\{\{a\}, \{b\}, \{a,b\}\},$$
$$\{\phi, \{a\}, \{b\}, \{a,b\}\}\}$$

$P(S) \cap P(P(S)) = \phi$

12/4/21 Lecture 1-B

Powerset Ques

Q.1 let x be any set and let $a, b \notin x$
If x has 'n' subsets and $Y = x \cup \{a, b\}$
then the number of subsets Y has

a) $4n$ b) $n+4$ c) $2n$ d) 2^{n+2}

let x has m elements

$|x| = m$

$Y = X \cup \{a, b\}$

$|Y| = |x| + |\{a,b\}|$

$|Y| = m + 2$

No. of subsets $= 2^m$

ATQ,

No. of subsets $= n$

$|P(x)| = n$

$2^m = n$ —①

$m = \log_2 n$

$|P(Y)| = 2^{\log_2 n + 2} = 2^{\log_2 n} \cdot 2^2$

$|P(Y)| = 4n$

$|P(Y)| = 2^{m+2} = 2^m \cdot 2^2$

$= 4n$ $(2^m = n)$

G: 2021

Q.2 Suppose that 'S' is a set with 10 elements, How many ordered pairs (A, B) are there such that A and B are subsets of S and $A \subseteq B$.

Each element of the set S appears in 3 ways so the total no. of ways to form ordered pairs is equal to 3^n.

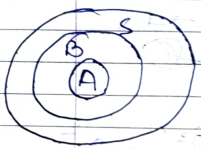
but here $n = 10$

$\therefore$ No. of ordered $= 3^{10}$

~~3 ways → absent~~

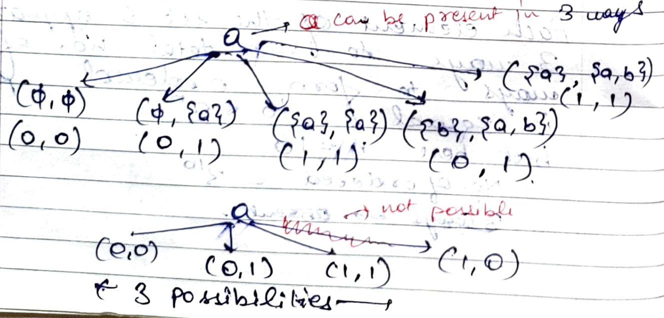
let $|S| = 2$, where $S = \{a, b\}$

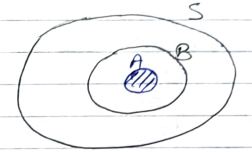
A	B
ϕ	ϕ
ϕ	$\{a\}$
ϕ	$\{b\}$
ϕ	$\{a, b\}$
$\{a\}$	$\{a\}$
$\{a\}$	$\{a, b\}$
$\{b\}$	$\{b\}$
$\{b\}$	$\{a, b\}$
$\{a, b\}$	$\{a, b\}$

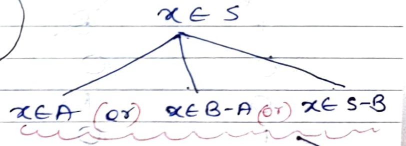


$$\left.\overset{0/1}{x_1}, \overset{0/1}{x_2}, \ldots \overset{0/1}{x_n}\right\} = 2^n$$

$$\left.\overset{0/1/2}{x_1}, \overset{0/1/2}{x_2}, \ldots, \overset{0/1/2}{x_n}\right\} = 3^n$$





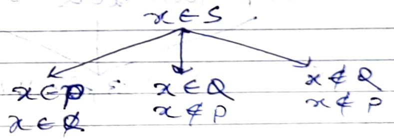


there are 10 elements

∴ No. of ordered pair = 3^{10}

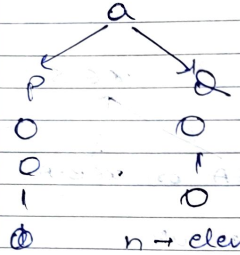
(as every element x has 3 possibilities)

Q let 'S' is a set with 'n' elements. If P, Q are subsets of 's', then how many ways we can find P, Q such that $P \cap Q = \phi$.



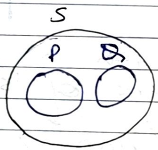
P	Q	$P \cap Q = \phi$
ϕ	ϕ	ϕ
ϕ	$\{a\}$	ϕ
ϕ	$\{b\}$	ϕ
ϕ	$\{a, b\}$	ϕ
$\{a\}$	ϕ	ϕ
$\{a\}$	$\{b\}$	ϕ
$\{b\}$	ϕ	ϕ
$\{b\}$	$\{a\}$	ϕ
$\{a, b\}$	ϕ	ϕ

$= 9 = 3^2$



n → element
3 → possibilities

∴ 3^n is the Ans



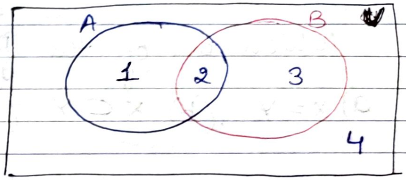
$\forall x \in S$

$x \in P$ (or) $x \in Q$ (or) $x \in S - \{P \cup Q\}$

3 possibilities = 3^n

Venn Diagram

- $U = \{1,2,3,4\}$
- $A \cup B = \{1,2,3\}$



1, 2, 3, 4 → region

- $B - A \equiv B - (A \cap B) = \{3\}$
- $A - B \equiv A - (A \cap B) = \{1\}$
- $A \cap B = \{2\}$

$$A \Delta B = \underbrace{(A-B) \cup (B-A)}_{\{1,3\}} \equiv (A \cup B) - (A \cap B)$$

$$= \{1,2,3\} - \{2\}$$

$$= \{1,3\}$$

here ~~AΔB~~ $A \Delta B$ is called symmetric difference of A, B.

G: 2005

① Let A, B and C be non empty sets and let $X = (A-B)-C$ and $Y = (A-C)-(B-C)$

Which one of the following is true

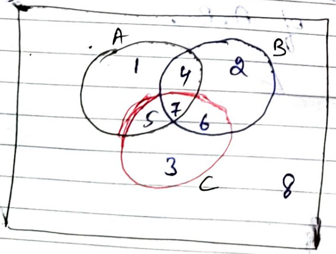
a) $X = Y$, (b) $X \subset Y$ (c) $Y \subseteq X$ (d) None of these

Ans ~~$(A-B)-C = A-(A\cap B)-(A\cap C)$~~

~~$(A-C)-(B-C) = A-(A\cap C)-(B-(B\cap C))$~~

~~$= A-(A\cap C)-B+(B\cap C)$~~

~~$= A-(A\cap B)-(A\cap C)+(B\cap C)$~~



$X = (A-B)-C$

$= (\{1, \cancel{4}, 5, \cancel{7}\} - \{2, \cancel{4}, \cancel{7}, 6\}) - \{5, 7, 6, 3\}$

$= \{1, \cancel{5}\} - \{\cancel{5}, 7, 6, 3\}$

$= \{1\}$

$Y = (A-C)-(B-C)$

$= \{1, \cancel{4}\} - \{2, \cancel{4}\}$

$= \{1\}$

$\therefore X = Y$

Option A ✓

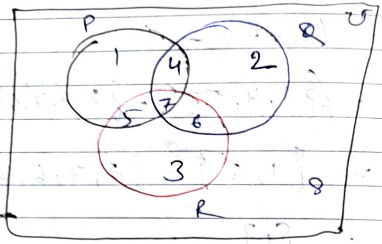
G: 2008

Q If P, Q, R are subsets of the universal set U then
$(P\cap Q\cap R) \cup (P^c \cap Q \cap R) \cup Q^c \cup R^c$ is

a) $Q^c \cup R^c$ b) $P \cup Q^c \cup R^c$

c) $P^c \cup Q^c \cup R^c$ d) U

$A^c = U - A$; Ex: $U = \{1, 2, 3, 4\}$
$A = \{2, 3\}$
$A^c = \{1, 4\}$



$P = \{1, 4, 5, 7\}$

$P^c = \{2, 6, 3, 8\}$

$Q = \{2, 4, 6, 7\}$

$Q^c = \{1, 5, 3, 8\}$

$R = \{5, 6, 7, 3\}$

$R^c = \{1, 4, 2, 8\}$

$P \cap Q \cap R = \{1,4,5,7\} \cap \{2,4,6,7\} \cap \{5,6,7,3\}$

$= \{7\}$

$(P^c \cap Q \cap R) = \{2,6,3,8\} \cap \{2,4,6,7\} \cap \{5,6,7,3\}$

$= \{6\}$

$Q^c = \{1,5,3,8\}$; $R^c = \{1,4,2,8\}$

$Q^c \cup R^c = \{1,5,3,8,4,2\}$

$(P \cap Q \cap R) \cup (P^c \cap Q \cap R) \cup Q^c \cup R^c$

$= \{1,5,3,8,4,2,7,6\}$

$= U$

D → Correct Answer

Q G: 2006

Let E, F and G be finite sets

Let $X = (E \cap F) - (F \cap G)$

$Y = (E - (E \cap G)) - (E - F)$

Which of the following is TRUE?

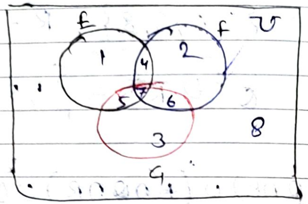
(A) $X \subset Y$ (B) $X \supset Y$ (C) $X = Y$

X is superset of Y

(D) $X - Y \neq \phi$ and

$Y - X \neq \phi$

Cancel
write the no. in the left set only



~~$X = \{1,5\} - \{$~~

$x = \{4,7\} - \{7,6\}$

$X = \{4\}$ ①

$Y = (E - \{5,7\}) - \{1,5\}$

$= \{1,4\} - \{1,5\}$

$Y = \{4\}$ ②

$X = Y$ (① = ②)

Q:1 Let $X = \{1,2,3,4, ----- ,2n\}$

and A, B are subset of 'X' such that

$A \Delta B = (A - B) \cup (B - A)$.

How many ways we can choose A, B such that

$A \Delta B = \{2, 4, 6, 8, 10 -- 2n\}$

Q2) Let U is a power of Set $S = \{1,2,3,4,5,6\}$

Let T' be the complement of T. for $T, R \in U$.

$\frac{T}{R}$ denote set of all elements which are in T but not in R

Which of the following is TRUE?

Ⓐ $\forall x \in U \ (|x| = |x'|)$.

Ⓑ $\exists x \in U \ \exists y \in U \ (|x| = 5, |y| = 5 \text{ and } x \cap y = \phi)$

Ⓒ $\forall x \in U, \forall y \in U \ (|x| = 2, |y| = 3 \ \& \ \frac{x}{y} = \phi)$

Ⓓ $\forall x \in U, \forall y \in U \ (\frac{x}{y} = \frac{y'}{x'})$

Ans-1 2^n

Ans-2 D option

Lecture -2

12/4/21

Sol 2

Ⓐ $\forall x \in U \,(|x| = |x'|)$ false

$(\because x' = U - x$

$\Rightarrow |x'| \neq |x|$

$S = \{\underbrace{1,2,3,4,5,6}_{6}\}$

$U = \{\phi, \{1\}, \overset{x}{\{2\}}, \{3\}, \overset{Y}{\{4\}}, \{5\}, \{6\},$
$\{1,2\}, \{1,3\}, \{1,4\}, \text{----}\}$

false

Ⓑ

$\exists x \in U, \exists Y \in U$

$|x| = 5, |Y| = 5 \;\&\; x \cap Y = \phi$

False $\quad |S| = 6$

So $|x| = 5, |Y| = 5 \;\&\; x \cap Y = \phi$
is not possible

Ⓒ $\forall x \in U, \forall Y \in U \left(|x| = 2 \;\&\; |Y| = 3, \frac{x}{Y} = \phi\right)$

False

$S = \{1,2,3,4,5,6\}$

$U = \{\phi, \{1\}, \{2\}, \text{----}, \{6\}, \overset{x}{\{1,2\}}, \text{----}$
$\underset{}{\overset{Y}{\{3,4,5\}}} \text{---} \}$

Now $\frac{x}{Y} = \{1,2\} - \{3,4,5\}$

$= \{1,2\} \neq \phi$

Ⓓ $\forall x \in U, \forall Y \in U \left(\frac{x}{Y} = \frac{Y'}{x'}\right)$

Now $\frac{x}{Y} = \frac{Y'}{x'} = \frac{U - Y}{U - x}$ $\quad$ [U → universal set]

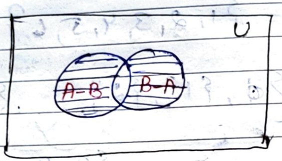
$x - Y = (U - Y) - (U - x)$

$x - Y = x - Y$ $\quad$ (TRUE)

Ans-1 $A \Delta B = (A - B) \cup (B - A)$

OR

$= (A \cup B) - (A \cap B)$



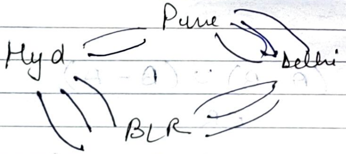
$\forall x \in X$

If 'x' is even → $\left[x \in (A-B) \cdot \text{or} \; x \in (B-A) \right]$

If 'x' is odd → $\left[x \in A \cap B \; (\text{or}) \; x \in U-(A \cup B) \right]$

$2^n \times \boxed{2^n}$ → It is compulsory to show that odd no. are at their defined place only

$= 2^{2n}$



Hyd → Delhi (Pune) → 6

" " (BLR) → 9

Hyd → Delhi = 9 + 6 = 15

Multiset

Multisets are unordered collection of elements where an element can occur as a member more than once.

$$A = \{\overset{\downarrow}{3a}, \overset{\uparrow}{4b}, 2c, d\}$$

multiplicity of 'b'

multiplicity of 'a'

here 'A' is called multiset

$B = \{4a, 2b, 4d\}$

here 'B' is called multiset

No. of transaction performed on particular bank by A & B

① $A = \{3a, 2b, 1c\} = \{3a, 2b, 1c, 0 \cdot d\}$

$B = \{2a, 4b, 3d\} = \{2a, 4b, 0 \cdot c, 3d\}$

$A \cup B = \{3a, 4b, 1 \cdot c, 3d\}$

$A \cap B = \{2a, 2b, 0 \cdot c, 0 \cdot d\} = \{2a, 2b\}$

$A - B = \{1 \cdot a, 1 \cdot c\}$.

$B - A = \{2b, 3d\}$

$A + B = \{5a, 6b, 1 \cdot c, 3 \cdot d\}$

Q How many number of multisets are possible with the element of set {a, b, c}?

A $\{a, bb, ccc\} = \{a, 2b, 3c\}$

$\{aaa, bbbb, ccccc\} = \{3a, 4b, 5c\}$

⋮

We can construct as many as multisets as we can repeat the element.

∴ Infinite sets are possible.

Relation

Cartisian Product :-

Let A, B are two sets

$A \times B = \{(x, y) / x \in A, y \in B\}$

$A = \{1, 2, 3\}, \quad B = \{p, q, r\}$

$A \times B = \{(1,p), (1,q), (1,r), (2,p), (2,q), (2,r), (3,p), (3,q), (3,r)\}$

$B \times A = \{(p,1), (p,2), (p,3), (q,1), (q,2), (q,3), (r,1), (r,2), (r,3)\}$

If $|A| = m$, $|B| = n$

then $|A \times B| = m \times n = mn$

Relation :-

let A, B are two sets

Any subset of $A \times B$ is called relation from A to B

If $R \subseteq A \times B$, then 'R' is called Relation from A to B

let $A = \{1, 2, 3\}$, $B = \{p, q\}$

$A \times B = \{(1,p), (1,q), (2,p) (2,q) (3,p) (3,q)\}$

$R_1 = \phi \subseteq A \times B$

$\therefore$ R_1 is a relation from A to B

$R_2 = \{(1,p)\} \subseteq A \times B$

$\therefore$ R_2 is a relation from A to B

$R_3 = \{(2,p), (3,q)\} \subseteq A \times B$

$\therefore$ R_3 is a relation from A to B

but $R_4 = \{(2,p) (3,r)\} \not\subseteq A \times B$

$\therefore$ R is not a Relation

Note:- If $|A| = m$, $|B| = n$ then

$|A \times B| = mn$

Total no. of Relations from A to B

$= $ Total no. of subsets of $A \times B$

$= |P(A \times B)|$

$= 2^{mn}$ $\left(\because \text{If } |S| = n, \; |P(S)| = 2^n\right)$

GATE

Q1) If 'A' is a set with 'n' elements then how many no. of relations are possible from A to A?

Ⓐ 2^n Ⓑ n^2 Ⓒ 2^{n^2} Ⓓ 2^{2n}

$A = \{1, 2, 3, \ldots, n\}$

$|A \times A| = \{ n \times n$ element

$|A \times A| = n^2$

No. of relations from A to A $= |P(A \times A)|$

$= 2^{n^2}$

C → correct option

Relation:-

let A, B are two sets

Any subset of $A \times B$ is called relation from A to B

If $R \subseteq A \times B$, then 'R' is called Relation from A to B.

Let $A = \{1, 2, 3\}$, $B = \{p, q\}$

$A \times B = \{(1,p), (1,q), (2,p) (2,q) (3,p) (3,q)\}$

$R_1 = \phi \subseteq A \times B$

∴ R_1 is a relation from A to B

$R_2 = \{(1,p)\} \subseteq A \times B$

∴ R_2 is a relation from A to B

$R_3 = \{(2,p), (3,q)\} \subseteq A \times B$

∴ R_3 is a relation from A to B

but $R_4 = \{(2,p) (3,r)\} \not\subseteq A \times B$

∴ R is not a Relation

Note:- If $|A| = m$, $|B| = n$ then

$|A \times B| = mn$

Total no. of Relations from A to B

= Total no. of subsets of $A \times B$

$= |P(A \times B)|$

$= 2^{mn}$ $\left(\because \text{ If } |S| = n, \; |P(S)| = 2^n \right)$

GATE

Q1) If 'A' is a set with 'n' elements then how many no. of relations are possible from A to A?

(A) 2^n (B) n^2 (C) 2^{n^2} (D) 2^{2n}

$A = \{1, 2, 3, \ldots, n\}$

$|A \times A| = n \times n$ element

$|A \times A| = n^2$

No. of relations from A to A $= |P(A \times A)|$

$= 2^{n^2}$

C → correct option

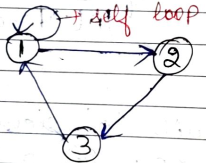
* Representation of Relation

$A = \{1, 2, 3\}$

$A \times A = \{(1,1), (1,2), (1,3) (2,1) (2,2), (2,3), (3,1) (3,2) (3,3)\}$

$R = \{(1,1)\ (1,2)\ (2,3)\ (3,1)\} \subseteq A \times A$

Diagraph :



(1, 2, 3 as vertices as it is given in A)

Matrix

$$M_R = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{array}$$

$A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (1,3) (2,1) (2,2), (2,3) (3,1) (3,2), (3,3)\}$

$(1,1) \rightarrow 1$ (That's why I came here)

$(2,2) \rightarrow 0$ (as (2,2) not present)

$$M_R = \begin{bmatrix} \underset{1}{(1,1)} & \underset{1}{(1,2)} & \underset{1}{(1,3)} \\ \underset{1}{(2,1)} & \underset{1}{(2,2)} & \underset{1}{(2,3)} \\ \underset{1}{(3,1)} & \underset{1}{(3,2)} & \underset{1}{(3,3)} \end{bmatrix}$$

(all present that's why 1)

Diagonal Relation

$A = \{1, 2, 3\}$

$\Delta_A = R = \{(1,1), (2,2), (3,3)\} \subseteq A \times A$

Diagonal Relation symbol $\Delta_A = \{(x, x) / \forall\, x \in A\}$

Complement of a Relation

$A = \{1, 2, 3\}$

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3) (3,1) (3,2) (3,3)\}$

$R = \{(1,2), (1,3), (2,3)\} \subseteq A \times A$

$R^c = (A \times A) - R$

$= \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\}$

(R is relation from A to A)

Inverse of a Relation

$R^{-1} = \{(b,a) / (a,b) \in R\}$

$A = \{1,2,3\}$

$R = \{(1,2),(1,3),(2,3)\}$

$R^{-1} = \{(2,1),(3,1),(3,2)\}$

#

let $A = \{1,2,3,4,----,n\}$

$|A| = n$

$$A\times A = \{\underbrace{(1,1),(2,2),\dots(n,n)}_{\text{'n' Diagonal elements}}\ \overbrace{\underbrace{(1,2)(2,1)(1,3)(3,1),\dots,(1,n)(n,1)}_{\hookrightarrow n^2-n}}^{\frac{n^2-n}{2}\text{ symmetric pairs}}\}$$

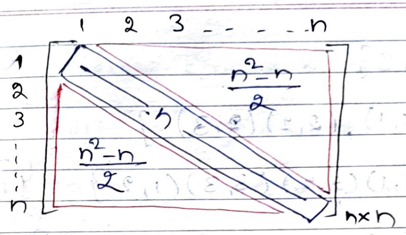
∴ $|A\times A| = n^2$

i) Total no. of elements $= n^2$

ii) No. of Diagonal elements $= n$

iii) No. of non-Diagonal elements $= n^2-n$

iv) No. of symmetric pairs $= \frac{n^2-n}{2}$



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13/4/21

TYPES OF RELATIONS

1) Reflexive Relation

Let 'R' is a relation on set 'A' $\forall x \in A$, if xRx (or) $(x,x) \in R$ then R is called Reflexive Relation on set 'A'

i) Smallest Reflexive Relation on 'A' $= \Delta_A$

ii) Largest Reflexive Relation on 'A' $= A\times A$

iii) If $|A| = n$, then how many no. of Reflexive Relation are on A $= 1\times 2^{n^2-n}$

In reflexive relation all diagonal elements must be present.

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$A = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2) (3,3)\}$

$R_2 = \{(1,1) (2,2) (3,3) (1,2\}$

$R_3 = \{(1,1) (2,2) (3,3) (1,3) (3,1)\}$

All are Reflexive Relation

$$A \times A = \{\overbrace{(1,1)(2,2) \ldots, (n,n)}^{'n'} \; \underbrace{\overset{1/0}{(1,2)} \overset{1/0}{(2,1)} \overset{1/0}{(1,3)}, \overset{1/0}{(3,1)} \ldots \overset{1/0}{(1,n)} \overset{1/0}{(n,1)}}_{n^2-n}\}$$

$$\boxed{1} \times \boxed{2^{n^2-n}}$$

→ non-diagonal may be present or absent

we have to choose all diagonal elements (So we can choose all elements in only 1 way)

Irreflexive Relation:-

$\forall x \in A$, $x \not R x$ (or) $(x, x) \notin R$

then 'R' is called Irreflexive

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i) Smallest = Empty Relation (i.e $R_1 = \phi$)

ii) Largest = $(A \times A) - \Delta_A$

iii) If $|A| = n$, then total no. of Irreflexive Relation on 'A' = $1 \times 2^{n^2-n}$

$= 1 \times 2^{n^2-n}$

$$A \times A = \{\underbrace{\overbrace{(1,1)(2,2) \ldots (n,n)}^{'n' \text{ elements}}}_{0 \quad 0 \quad 0} \; \overbrace{\overset{}{\underset{0/1}{(1,2)}} \underset{0/1}{(2,1)} \underset{0/1}{(1,3)} \underset{0/1}{(3,1)} \ldots \underset{0/1}{(1,n)} \underset{0/1}{(n,1)}}^{n^2-n}\}$$

$$\boxed{1} \times 2^{n^2-n}$$

All diagonal elements are absent only in 1 way

→ either non diagonal elements are present or absent.

Symmetric Relation

→ All diagonal elements are symm. by default.

Let 'A' is a non empty set

Whenever $(a,b) \in R$ then $(b,a) \in R$, $\forall a, b \in A$

i) Smallest = ϕ

ii) Largest = $A \times A$

iii) If $|A| = n$ then how many no. of symmetric relations are possible on A

$2^n \times 2^{\frac{n^2-n}{2}}$

$$A \times A = \{\underbrace{\overset{0/1}{(1,1)}\overset{0/1}{(2,2)}\ldots\overset{0/1}{(n,n)}}_{\text{'n' elements}},\ \underbrace{\overset{0/1}{\overbrace{(1,2)(2,1)}}\overset{0/1}{\overbrace{(1,3)(3,1)}},\ldots\overset{0/1}{\overbrace{(1,n)(n,1)}}}_{\frac{n^2-n}{2}\text{ symmetric pairs}}\}$$

$$2^n \times 2^{\frac{n^2-n}{2}}$$

Antisymmetric Relation :-

$\forall\ a,b \in A$

If $(a,b) \in R$ and $(b,a) \in R$

then $a = b$

Note:- If $(a,b) \in R$ and $(b,a) \in R$ such that $a \neq b$

then 'R' is not Antisymmetric Relation

$A = \{1,2,3\}$

$R_1 = \{(1,2),(2,1)\}$ not Antisymmetric

$R_2 = \{(1,2)\}$ Antisymmetric

$R_3 = \phi$ Antisymmetric

i) Smallest $= \phi$

ii) If $|A| = n$ then how many no. of elements are there in largest Antisymmetric Relation

$$= n + \frac{n^2-n}{2}$$

Let $A = \{1,2,3\}$

$R = \{(1,1),(2,2)(3,3)\ (1,2)\ (1,3)\ (2,3)\}$

$$\begin{array}{c|ccc|l} & 1 & 2 & 3 & \\ \hline 1 & 1 & 1 & 1 & \\ 2 & 0 & 1 & 1 & \longrightarrow \frac{n^2-n}{2} \\ 3 & 0 & 0 & 1 & \longrightarrow n \end{array}$$

$$= n + \frac{n^2-n}{2} = \frac{n(n+1)}{2}$$

iii) If $|A| = n$ then how many no. of Antisymmetric Relation are on 'A'

$$A \times A = \{\underbrace{\overset{1/0}{(1,1)}\overset{1/0}{(2,2)}\ldots\overset{1/0}{(n,n)}}_{n}\underbrace{\overset{\begin{smallmatrix}0 & 1\\ 1 & 0\\ 0 & 0\end{smallmatrix}}{\overbrace{(1,2)(2,1)}}\overset{\begin{smallmatrix}0 & 1\\ 1 & 0\\ 0 & 0\end{smallmatrix}}{\overbrace{(1,3)(3,1)}}\ldots(1,n)(n,1)}_{\frac{n^2-n}{2}\text{ symmetric pairs}}\}$$

$$= 2^n \times 3^{\frac{n^2-n}{2}}$$

Golden Rules

~~Diag~~	Diagonal	Single non Diagonal ordered pair
Symmetric	✓	✗
Antisymmetric	✓	✓
Transitive	✓	✓
Asymmetric	Absent	✓

Asymmetric Relation :-

- Whenever $(a,b) \in R$ then $(b,a) \notin R$ $\forall a,b \in A$
- All diagonal elements are always absent
- Every Asymmetric Relation is Antisymmetric
- If $|A| = n$ then total no. of Asymmetric relation on A

$$= 1 \times 3^{\frac{n^2-n}{2}}$$

* Every Asymmetric Relation is Antisymmetric

- Smallest $= \phi$
- GATE :
  If $|A| = n$ then how many no. of elements in largest Asymmetric Relation

$$= \frac{n^2 - n}{2}$$

(As Diagonal elements are absent)

Transitive

Whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ $\forall a,b,c \in A$

Note: whenever $(a,b) \in R$ and $(b,c) \in R$ such that $(a,c) \notin R$ then 'R' is called not Transitive

$A = \{1, 2, 3\}$

$R_1 = \phi$

$R_2 = \{(1,2)\}$

$R_3 = \{(1,1)\ (2,2)\ (3,3)\}$

$R_4 = \{\overset{b\,c}{(1,2)}\ \overset{a\,b}{(2,1)}\ (1,1)\}$ ✗ (2,2 missing)

$R_5 = \{(1,2)\ (2,1)(1,1)\ (2,2)\}$

i) Smallest $= \phi$

ii) largest $= A \times A$

~~Navy~~ formula for no. of Transitive

Equivalence Relation

A relation 'R' on set 'A' is called equivalence Relation if 'R' is

i) Reflexive ii) Symmetric iii) Transitive

i) Smallest : Δ_A

ii) Largest : $A \times A$

$A = \{1, 2, 3\}$

$R_1 = \phi \subseteq A \times A$

$R_2 = \{(1,1)(1,2)(1,3)(2,1)(2,2)(2,3)(3,1)(3,2)(3,3)\} \subseteq A \times A$

$R_3 = \{(1,1)(2,2)(3,3)\} = \Delta_A \subseteq A \times A$

$R_4 = \{(1,1)(1,2)(2,1)\}$

$R_5 = \{(1,1)(2,2)(1,3)(3,1)(2,3)\} \subseteq A \times A$

$R_6 = \{(1,2)(2,3)(3,2)\} \subseteq A \times A$

$R_7 = \{(1,2)(2,1)(1,3)(3,1)(2,3)(3,2)\} \subseteq A \times A$

$R_8 = \{(1,2)\}$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
Reflexive	✗	✓	✓	✗	✗	✗	✗	✗
Irreflexive	✓	✗	✗	✗	✗	✓	✓	✓
Symmetric	✓	✓	✓	✓	✗	✗	✓	✗
Antisymmetric	✓	✗	✓	✗	✗	✗	✗	✓
Asymmetric	✓	✗	✗	✗	✗	✗	✗	✓
Transitive	✓	✓	✓	✗	✗	✗	✗	✓
Equivalence	✗	✓	✓	✗	✗	✗	✗	✗

Q.1) If $A = \{1, 2\}$ then how many no. of equivalence Relations are possible on A ?

S ~~Symmetric~~

$A \times A = \{$ ~~ϕ~~ $(1,1)\ (1,2)\ (2,1)\ (2,2)\}$

R_1 (Reflexive) $= 1 \times 2^2$

Symmetric $= 2^2 \times 2^1$

Transitive $=$ ~~2^2~~ 6

Total no $= 4 + 8 + 6 = 18$

Completely wrong

$R_1 = \{(1,1) (2,2)\}$

$R_2 = \{(1,1) (2,2) (1,2) (2,1)\}$

Only 2 equivalence relations are present.

Q.2) If $A = \{1,2,3\}$ then how many no. of equivalence Relation on A.

d. $R_1 = \{(1,1) (2,2) (3,3)\}$

~~$R_2 = \{(1,1)(2,2)(3,3)(1,2)(2,1)\}$~~

$R_2 = \{(1,1)(2,2)(3,3)(1,2)(2,1)(1,3)(3,1)(2,3)(3,2)\}$

$= A \times A$

$R_3 = \{(1,1) (2,2) (3,3) (1,2) (2,1)\}$

$R_4 = \{(1,1) (2,2) (3,3) (1,3) (3,1)\}$

$R_5 = \{(1,1) (2,2) (3,3) (2,3) (3,2)\}$

$R_6 = \{(1,1) (2,2) (3,3) (\overset{b}{1},\overset{c}{2}) (2,1) (1,3) (\overset{a}{3},\overset{b}{1})\}$

→ not Transitive ↑ (3,2) missing

so not equivalence Relation

Total no. of equivalence Relation = 5

---

* $|A| = n$, then total no. of equivalence Relations $= B_n$

where 'B_n' is Bell number

n	No. of Equivalence Relation
1	$B_1 = 1$
2	$B_2 = 2$
3	$B_3 = 5$
4	$B_4 = 15$
5	$B_5 = 52$
6	$B_6 = 203$

Gate 1999

Q. The number of binary relation on a set with n element is

a) n^2 b) 2^n

c) 2^{n^2} d) None of the above

$|A| = n$

$|A \times A| = n^2$

$\therefore$ No. of Relations $= |P(A \times A)|$

$= 2^{n^2}$

GATE:

Q. The number of different n×n symmetric matrices with each element being either 0 or 1 is
(Note :- power (2, n) is same as 2^n) . ___

a) power (2, n) b) power (2, n^2)

c) power (2, $(n^2+n)/2$) d) power (2, $(n^2-n)/2$)

Sol = Different n×n symmetric matrices means no. of ~~ss~~ symmetric relations

$$= 2^n \times 2^{\frac{n^2-n}{2}}$$

$$= 2^{n + \frac{n^2-n}{2}}$$

$$= 2^{\frac{n^2+n}{2}}$$

C → correct option

GATE - 09

Q Consider the binary relation $R = \{(x,y), (x,z), (z,x), (z,y)\}$ on the set $\{x, y, z\}$.

Which one of the following is TRUE ?

a) R is symm but Not Antisymm.
b) R is not symm but antisymm.
c) R is both symm and Antisymm.
d) R is neither symm nor Antisymm.

$$M = \begin{array}{c|ccc} & x & y & z \\ \hline x & 0 & 1 & 1 \\ y & 0 & 0 & 0 \\ z & 1 & 1 & 0 \end{array}$$

Not Reflexive   Irreflexive

- If $M^T = M$ then M is symm
(here $M^T \neq M$ so not symm)

- not Antisymmetric $\underbrace{(x,z)\ (z,x)}_{\text{both present}}$
↓
Not Asymmetric

every Asymmetric is Antisymmetric Relation.
(D - option)

Q what is the possible number of reflexive relation on a set of 5 element

a) 2^{10} b) 2^{15} c) 2^{20} d) 2^{25}

Sol $= 1 \times 2^{n^2-n}$

$= 1 \times 2^{25-5}$

$= 2^{20}$

(C - option)

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GATE: 98

Q The binary relation

$R = \{(1,1)\}, (2,1), (2,2), (2,3) (2,4) (3,1) (3,2) (3,3) (3,4)\}$

on the set $A = \{1, 2, 3, 4\}$ is

a) Ref, symm & transitive

b) Neither ref, nor irref but trans

c) Irref, symm & trans

d) Irreflexive & Antisymm

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

Not Reflexive (all diag. elements not present)

Not irreflexive &

$M^T \neq M$ (not symm)

Not Antisymm ((2,3) (3,2) both are true)

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Not Asymm (Diagonals elements present)

&

or since it is not Antisymm therefore it can't be Asymm also.

(B → option)

GATE - 02

Q The binary relation $S = \phi$ (empty set) on set $A = \{1, 2, 3\}$ is

a) Nether reflexive nor symmetric
b) Symmetric & reflexive
c) Transitive & reflexive
d) Transitive & Symm

~~d~~ d → option

Q GATE -15- set 3

Let R be a relation on the set of ordered pairs of positive integers such that $\{(p, q), (r, s)\} \in R$ if and only if $p - s = q - r$. which one of the following is true about R?

a) Both reflexive & Symm
b) Reflexive nor symm.
c) Not reflexive but symm
d) Neither reflexive ~~Neither~~ nor symm

Let $A = \{(a,b)(p,q)(r,s) - - - \}$

$\{((p,q),(r,s)) \in R$ iff $p-s = q-r$

- Reflexive:- $\forall (p,q) \in A$, we have to show that $((p,q),(p,q)) \in R$

If $(p,q) R (p,q)$ then

$p - q = q - p$

(which is false)

$\therefore ((p,q)(p,q)) \notin R$

$\therefore$ not reflexive

$\rightarrow$ $(p,q) R (r,s)$

$\Rightarrow$ $p - s = q - r$

$\Rightarrow$ $s - p = r - q$

$\Rightarrow$ $r - q = s - p$

$(r,s) R (p,q)$

$\therefore$ It is symmetric

GATE - 96

Q Let R be a non empty relation on a collection of sets defined by $A R B$ if and only if $A \cap B = \phi$

Then, (pick the true statement)

a) R is reflexive and transitive

b) R is symmetric and not transitive

c) R is an equivalence relation

d) R is not reflexive and not symmetric

$S = \{ \underbrace{\{1,2\}}_{A}, \underbrace{\{3,4\}}_{B}, \underbrace{\{2,5\}}_{C}, \ldots \}$

$A R B$ iff $A \cap B = \phi$

i) Reflexive : $\forall A \in S$, we have to prove

$A R A$

But $A \cap A = \phi$ is not possible $\forall A \in S$

$\therefore A \not R A$

$\therefore$ not Reflexive

ii) Symmetric

let ARB

$\Rightarrow A \cap B = \phi$

$\Rightarrow B \cap A = \phi$

BRA

$\therefore$ symmetric

If $A = \{1, 2\}$
$B = \{3, 4\}$
$C = \{2, 5\}$

Clearly $A \cap B = \phi$
$B \cap C = \phi$

$A \cap C \neq \phi$

$\therefore A \not R C$

* B option is correct

* $S = \{1,2,3,4\}$ → set of elements

* $S = \{(a,b)(c,d)\}$ → set of ordered pair

* $S = \{\{1,2\},\{3,4\},\{3\}\}$ → set of sets

35:50

Q Let N be the set of all positive integers and R be a relation on N defined as follows:

Definition of R: for all $a, b \in N$, $(a,b) \in R$ iff $\frac{a}{b} = 2^i$ for some integer $i \geq 0$

Which of the following is not TRUE

a) R is reflexive
b) R is Antisymmetric
c) R is Transitive
d) R is symmetric

Sol $(a,b) \in R$ iff $\frac{a}{b} = 2^i$ $i \geq 0$

Ref :- $(a,a) \in R$

$\Rightarrow \frac{a}{a} = 2^0$

$\therefore$ Ref ✓

Antisymm:

let $aRb \Rightarrow \frac{a}{b} = 2^i$

and $bRa \Rightarrow \frac{b}{a} = 2^i$

$$\frac{a}{b} = \frac{b}{a}$$

$\Rightarrow a^2 - b^2 = 0$

$\Rightarrow (a-b)(a+b) = 0$

$a - b = 0 ; \quad a + b \neq 0$

$a = b$

Antisymm ✓

Symm :-

Let aRb

$\Rightarrow \frac{a}{b} = 2^i$

$\Rightarrow \frac{b}{a} = \frac{1}{2^i}$

$\Rightarrow b \not R a \quad (i < 0)$

Trans :-

Let aRb and bRc

$\Rightarrow \frac{a}{b} = 2^i , \quad \frac{b}{c} = 2^j$

Now $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c}$

$= 2^i \times 2^j$

$\frac{a}{c} = 2^{i+j} \quad \Rightarrow aRc$

Relationship between Reflexive and Irreflexive

Let $|A| = n$

i) Total no. of Relations on $A = 2^{n^2} = U$

ii) $n(Ref) = 1 \times 2^{n^2-n}$, $n(Irr) = 1 \times 2^{n^2-n}$

Q No. of Relations which are

a) Reflexive and Irreflexive $\Rightarrow n(Ref \cap Irr)$

b) Reflexive but not Irreflexive

$\Rightarrow n(Ref - Irr) = n(Ref) - n(Ref \cap Irr)$

c) Irreflexive but not Reflexive

$\Rightarrow n(Irr - Ref) = n(Irr) - n(Ref \cap Irr)$

d) Neither Reflexive nor Irreflexive

$\Rightarrow n(\overline{Ref \cup Irr}) = U - n(Ref \cup Irr)$

e) Either Reflexive or Irreflexive

$\Rightarrow n(Ref \cup Irr) = n(Ref) + n(Irr) - n(Ref \cap Irr)$

$$\frac{a}{b} = \frac{b}{a}$$

$$\Rightarrow a^2 - b^2 = 0$$

$$\Rightarrow (a-b)(a+b) = 0$$

$a - b = 0$; $a + b \neq 0$

$a = b$

Antisymm ✓

Symm :-

let aRb

$$\Rightarrow \frac{a}{b} = 2^{i}$$

$$\Rightarrow \frac{b}{a} = \frac{1}{2^{i}}$$

$$\Rightarrow b \not R a \quad (\because i < 0)$$

Trans :-

let aRb and bRc

$$\Rightarrow \frac{a}{b} = 2^{i}, \quad \frac{b}{c} = 2^{j}$$

Now $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c}$

$$= 2^{i} \times 2^{j}$$

$$\frac{a}{c} = 2^{i+j} \Rightarrow aRc$$

~~B~~ D option correct

Relationship between Reflexive and Irreflexive

Let $|A| = n$

i) Total no. of Relations on $A = 2^{n^2} = U$

ii) $n(\text{Ref}) = 1 \times 2^{n^2 - n}$, $n(\text{Irr}) = 1 \times 2^{n^2 - n}$

Q No. of Relations which are

a) Reflexive and Irreflexive $\Rightarrow n(\text{Ref} \cap \text{Irr})$

b) Reflexive but not Irreflexive

$\Rightarrow n(\text{Ref} - \text{Irr}) = n(\text{Ref}) - n(\text{Ref} \cap \text{Irr})$

c) Irreflexive but not Reflexive

$\Rightarrow n(\text{Irr} - \text{Ref}) = n(\text{Irr}) - n(\text{Ref} \cap \text{Irr})$

d) Neither Reflexive nor Irreflexive

$\Rightarrow n(\text{Ref} \cup \text{Irr}) = U - n(\text{Ref} \cup \text{Irr})$

e) Either Reflexive or Irreflexive

$\Rightarrow n(\text{Ref} \cup \text{Irr}) = n(\text{Ref}) + n(\text{Irr}) - n(\text{Ref} \cap \text{Irr})$

14/4/21 Lecture - 4

	$1\times 2^{n^2-n}$	$2^n\times 2^{\frac{n^2-n}{2}}$	$2^n\times 3^{\frac{n^2-n}{2}}$	$1\times 3^{\frac{n^2-n}{2}}$
ϕ	Irr	Sym	Anti	Asy
Ref : $1\times 2^{\frac{n^2-n}{2}}$	0	$1\times 2^{\frac{n^2-n}{2}}$	$1\times 3^{\frac{n^2-n}{2}}$	0
Irr : $1\times 2^{n^2-n}$	✗	$1\times 2^{\frac{n^2-n}{2}}$	$1\times 3^{\frac{n^2-n}{2}}$	$1\times 3^{\frac{n^2-n}{2}}$
Sym : $2^n\times 2^{\frac{n^2-n}{2}}$	✗	✗	$2^n\times 1$	1×1
Anti : $2^n\times 3^{\frac{n^2-n}{2}}$	✗	✗	✗	$1\times 3^{\frac{n^2-n}{2}}$

$\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix} \to$ 3 possibility

$A\times A = \{(1,1)(2,2)\ldots(n,n), (1,2)(2,1)(1,3)(3,1)\ldots(1,n)(n,1)\}$

$\{\begin{smallmatrix}1\\0\end{smallmatrix} \quad \begin{smallmatrix}1\\0\end{smallmatrix}\} \to$ 2 possibility (Common is (0,0))

Sym : $2^n \times 2^{\frac{n^2-n}{2}}$

Anti : $2^n \times 3^{\frac{n^2-n}{2}}$

Sym ∩ Anti : $2^n \times 1$

Sym : $2^n \times 2^{\frac{n^2-n}{2}} \longrightarrow \{(1,1), (0,0)\}$

Asy : $1 \times 3^{\frac{n^2-n}{2}} \longrightarrow (0,1), (0,0), (1,0)$

(Sym ∩ Asy) : 1×1

If $|A| = n$, Total no. of Relations $v = 2^{n^2}$

$n(Ref) = 1\times 2^{n^2-n}$, $n(Irr) = 1\times 2^{n^2-n}$

From prev. page $n(Ref \cap Irr) = 0$

1) $n(Ref \cup Irr) = (1\times 2^{n^2-n}) + (1\times 2^{n^2-n}) - 0$

2) $n(Ref - Irr) = (1\times 2^{n^2-n}) - 0$

3) $n(Irr - Ref) = (1\times 2^{n^2-n}) - 0$

4) $n(\overline{Ref \cup Irr}) = 2^{n^2} - [(1\times 2^{n^2-n}) + (1\times 2^{n^2-n}) - 0]$

Relationship between Symmetric And Antisymmetric

1) $n(Sym \cup Anti) = \left(2^n \times 2^{\frac{n^2-n}{2}}\right) + \left(2^n \times 3^{\frac{n^2-n}{2}}\right) - (2^n \times 1)$

2) $n(Sym - Anti) = \left(2^n \times 2^{\frac{n^2-n}{2}}\right) - (2^n \times 1)$

3) $n(Anti - Sym) = \left(2^n \times 3^{\frac{n^2-n}{2}}\right) - (2^n \times 1)$

4) $n(\overline{Sym \cup Anti}) = 2^{n^2} - \left[\left[2^n \times 2^{\frac{n^2-n}{2}}\right] + \left[2^n \times 3^{\frac{n^2-n}{2}}\right] - (2^n \times 1)\right]$

Q Let A = {1, 2, 3}. Number of relation on A which are neither reflexive nor irreflexive but symmetric is

$\{(1,1)(2,2)(3,3),(1,2)(2,1)(1,3)(3,1) \ldots (2,3)(3,2)\}$

$\times 1 \rightarrow (2^n - 2)$

$= (2^n - 2)\left(2^{\frac{n^2-n}{2}}\right)$

↳ as it is

$= 48 \quad (n=3)$

Q Let A = {1, 2}. No. of relations on A which are reflexive and symmetric, but not transitive is

{(1,1)(2,2)} → ref, symm, transitive

{(1,1)(2,2)(1,2)(2,1)} → ref, sym, Tran

∴ No case in case of n = 2 where relation is ref, symm & Trans.

∴ ⓪ (Answer)

∪ ∩	Ref R₂	Irr R₂	Sym R₂	Anti R₂	Asy R₂	Tran R₂	Equivalence R₂	Partial Order R₂
Ref R₁	✓ ✓							
Irr R₁		✓ ✓						
Sym R₁			✓ ✓					
Anti R₁				✗ ✓				
Asy R₁					✗ ✓			
Tran R₁						✗ ✓		
Equivalence R₁							✗ ✓	
Partial Order R₁								✗ ✓

A = {1, 2, 3}

$R_1 = \{(1,2)\}$ ← Asy, Anti, Tran

$R_2 = \{(2,1)\}$ ← Asy, Anti, Tran

$R_1 \cup R_2 = \{(1,2)(2,1)\}$ → not Anti
↳ not Asy
↳ not Tran

Partial Order :-
A relation 'R' on set 'A' which satisfies

i) Reflexive

ii) Antisymmetric

iii) Transitive

GATE - 87.

Q. State whether the following statements ~~are~~ are TRUE or FALSE.

a) The union of two equivalence relations is also an equivalence relation

b) The intersection of two equivalence relations is also an equivalence relation

Ans a) → FALSE

b) → TRUE

GATE 97

Q. The number of equivalence relations on the set {1, 2, 3, 4} is

a) 15 b) 16 c) 24 d) 4

A 15 → a option

BELL NUMBER

Q. Suppose A is a finite set with n elements. The number of elements in the largest equivalence relation of A is.

a) n b) n^2 c) 1 d) n+1

A B → n^2 (A × A) & largest equivalence relation

GATE - 07

Q. Let S be a set of n elements. The no. of ordered pairs in the largest and the smallest equivalence relations on S are

a) n and n b) n^2 and n

c) n^2 and 0 d) n and 1

largest equivalence relation = A × A

$= |A \times A| = n^2$

smallest equivalence relation = Δ_A

$= |\Delta_A| = n$

B option ✓

GATE - 01

Q Consider the following relations:

→ $R_1(a,b)$ iff $(a+b)$ is even over the set of integers

→ $R_2(a,b)$ iff $(a+b)$ is odd over the set of integers

→ $R_3(a,b)$ iff $a \cdot b > 0$ over the set of non zero rational number.

→ $R_4(a,b)$ iff $|a-b| \leq 2$ over the set of natural numbers.

Which of the following statements is correct

a) R_1 & R_2 are equivalence relation, R_3 and R_4 are not eq. relt^n

b) R_1 and R_3 are eq. relt^n, R_2 & R_4 are not eq. relation

c) R_1 and R_4 are eq. relation, R_2 and R_3 are not

d) R_1, R_2, R_3 and R_4 are all eq. relation.

a) Ref: $\forall a \in \mathbb{Z}$

$a+a = 2a$ (even)

$\therefore (a,a) \in R_1$ ($\because R_1$ is ref)

b) Sym: $\forall a, b, \in \mathbb{Z}$

Let $a R_1 b$

$\Rightarrow a+b = $ even

$\Rightarrow b+a = $ even

$\Rightarrow b R_1 a$ ($\because R_1$ is symm)

c) Trans: let $a R_1 b$, $b R_1 c$

$\Rightarrow a+b = 2n$ ①; $b+c = 2m$ ②

① + ②

$a+c = 2(m+n-b)$

$a+c = $ even

$a R_1 c$

R_1 → equivalence relation

R_3

$a \cdot a > 0$ (Ref)

$a \cdot b > 0$ (Symm)

$\therefore b \cdot a > 0$

$a \cdot b > 0$

$b \cdot c > 0$ (Trans)

$a \cdot c > 0$

R_4 → Ref, symm, but not Transitive

$(1, 3)$ $(3, 5)$

$|1-3| \leq 2$

$|3-5| \leq 2$

$|1-5| \nleq 2$

Equivalence Class :-

Given any set A and an equivalence relation R on A, the equivalence class [x] of each element of x of A is defined by

$[x] = \{y / (x,y) \in R\}$

Let $A = \{1, 2, 3, 4, 5\}$

$R = \{(1,1)(2,2)(3,3)(4,4)(5,5)(1,3)(3,1)(2,5)(5,2)\}$

$[1] = \{1, 3\}$

$[2] = \{2, 5\}$

$[3] = \{3, 1\}$.

$[4] = \{4\}$

$[5] = \{5, 2\}$

$\therefore$ $[1] = [3]$; $[2] = [5]$.

$[1], [2], [4]$ → Disjoint set (no common element)

$P = \{ \underbrace{\{1,3\}, \{2,5\}, \{4\}}_{} \}$ is partition of given set 'A'

Union will give the original set 'A'.

Partition :-

A partition of S is a sub division of S into non overlapping, non empty subsets such that

$A_1 \cup A_2 \cup A_3 \ldots \cup A_n = S$ and

$A_i \cap A_j = \phi$ if $A_i \neq A_j$

Example:

$S = \{a, b, c, d, e\}$

$P_1 = \{\underbrace{\{a, e\}}_{A_1}, \underbrace{\{b, c\}}_{A_2}, \underbrace{\{d\}}_{A_3}\}$

① $A_1 \cup A_2 \cup A_3$
$= \{a, b, c, d, e\}$
$= S$

② $A_1 \cap A_2 = \phi$
$A_1 \cap A_3 = \phi$
$A_2 \cap A_3 = \phi$
∴ P_1 is partition of S

$P_2 = [\{a, e, b\} \ \{b, c, d\}]$

$P_3 = \{\{a, b\} \ \{c, e\}\}$

$\rightarrow A_i \cap A_j \neq \phi$

P_2 is not partition of 's'

P_3 is not partition of 's'.

↓ Union is not giving original set

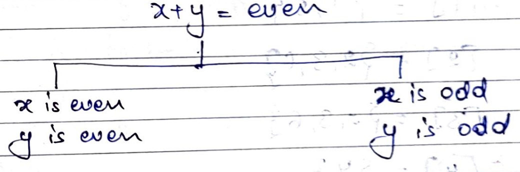
Q A relation R is defined on the set of integers as $x R y$ iff $(x+y)$ is even. which of the following statements is TRUE?

a) R is not an equivalence relation

b) R is an eq. relt^n having 1 eq. class

c) R " " " " " 2 eq. class

d) " " " " " " 3 eq. class



• if 'x' is even then
$[x] = \{\text{all even numbers}\}$

• if 'x' is odd then
$[x] = \{\text{all odd numbers}\}$

1, 2, 4 ∴ 3

Q let R be the following equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(1,1), (1,5), (2,2), (2,3), (2,6), (3,2), (3,3), (3,6), (4,4), (5,1), (5,5), (6,2), (6,3), (6,6)\}$

The no. of distinct equivalence classes on A with respect to R is ______.

$[1] = \{1,5\}$

$[2] = \{2,3,6\}$

$[3] = \{2,3,6\}$

$[4] = \{4\}$

$[5] = \{1,5\}$

$[6] = \{2,3,6\}$

[1], [2], [4] are distinct equivalence classes.

∴ Total no = 3

Q Let R be an equivalence relation on a set A. For any two elements $x, y \in A$, consider the following statements,

$S_1 : [x] = [y]$

$S_2 : [x] \cap [y] = \{ \}$

where, [x] and [y] are equivalence classes of x and y respectively.

Which of the following is TRUE?

a) S_1 is true and S_2 is false

b) S_1 is false and S_2 is true

c) Both S_1 and S_2 are true

d) Either S_1 or S_2 is true

Sol equivalence class when compare it can be equal or disjoint only,

So any one can at a time

∴ Either S_1 or S_2 is true

D → correct option.

Q Let $S = \{1, 2, \ldots 10\}$

Define the following four sets as

$P_1 = \{\{1,3,8\}, \{2,4,6\}, \{5,7,10\}, \{9\}\}$

$P_2 = \{\{7,4,3,8\}, \{1,5,10,3\}, \{2,6\}\}$

$P_3 = \{\{1,5,9\}, \{2,10,4,7\}, \{8,3,6\}\}$

$P_4 = \{\{4,2\}, \{3,8\}, \{6\}, \{10,7\}, \{1\}, \{5\}, \{9\}\}$

Which of the sets is not a partition of S?

a) P_1 b) P_2 c) P_3 d) P_4

P_1 ✓ P_2 ✗

P_3 ✓ P_4 ✓

b → Correct option

✓ → partitions of S

✗ → not partition of S

Suppose we have two partition of set 'S' say they are

$A = \{A_1, A_2, \ldots\}$

$B = \{B_1, B_2, \ldots\}$

We can say A is refinement of B if every set A_i of A we have to find B_j of B such that $A_i \subseteq B_j$

$A_i \subseteq B_j$

Q Let $S = \{1, 2, \ldots, 10\}$

Define the following four sets as:

$P_1 = \{\underbrace{\{1,3,8\}}_{B_1}, \underbrace{\{2,4,6\}}_{B_2}, \underbrace{\{5,7,10\}}_{B_3}, \underbrace{\{9\}}_{B_4}\}$

$P_2 = \{\{7,4,3,8\}, \{1,5,10,3\}, \{2,6\}\}$

$P_3 = \{\{1,5,9\}, \{2,10,4,7\}, \{8,3,6\}\}$

$P_4 = \{\underbrace{\{4,2\}}_{A_1}, \underbrace{\{3,8\}}_{A_2}, \underbrace{\{6\}}_{A_3}, \underbrace{\{10,7\}}_{A_4}, \underbrace{\{1\}}_{A_5}, \underbrace{\{5\}}_{A_6}, \underbrace{\{9\}}_{A_7}\}$

Which of the following is TRUE

a) P_4 is a refinement of both P_2 and P_3

b) P_4 " " " " P_1 and P_2

c) P_4 " " " " P_1, P_2 and P_3

d) P_4 " " " " P_1 and P_3

$A_1 \subseteq B_2$, $A_2 \subseteq B_1$

$A_3 \subseteq B_2$ $A_5 \subseteq B_1$

$A_4 \subseteq B_3$ $A_6 \subseteq B_3$

$A_7 \subseteq B_4$

∴ P_4 is refinement of P_1

P_2 → not partition

P_3 → satisfying the condition for refinement

∴ D → correct option

16/4/21 Lecture -5A

Closure

i) Reflexive Closure ($R^* = \Delta_S \cup R$)

$S = \{1, 2, 3, 4, 5\}$ $R = \{(1,2)(1,3)\}$

$R^* = \{(1,1)(2,2)(3,3)(1,2)(1,3)(4,4)(5,5)\}$

Smallest Reflexive Relation which contains given relation R

Eg 2: $S = \{1, 2, 3\}$, $R = \{(2,3)(4,5)\}$

$R^* = \{(1,1)(2,2)(3,3)(2,3)(4,5)\}$

ii) Transitive Closure

Smallest Transitive relation which contain given R

① $S = \{1, 2, 3\}$ $R = \{(1,2)\}$

$R^{\#} = \{(1,2)\}$

② $S = \{1, 2, 3\}$ $R = \{(1,2)(2,1)\}$

$R^{\#} = \{(1,2)(2,1)(1,1)(2,2)\}$

Since after (1,2)(2,1) the relation is not Transitive so we have to make it transitive by including necessary ordered pairs

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iii) Symmetric Closure ($R^+ = R \cup R^{-1}$)

Smallest symmetric relation which contain given relation R is called symmetric closure of R.

① $S = \{1,2,3\}$ $R = \{(1,2)\}$

$R^+ = \{(1,2)\ (2,1)\}$

② $S = \{1,2,3\}$ $R = \{(1,2)\ (1,3)\ (2,3)\}$

$R^+ = \{\underbrace{(1,2)\ (1,3)\ (2,3)}_{R},\ \underbrace{(2,1)\ (3,1)\ (3,2)}_{R^{-1}}\}$

$R \quad \cup \quad R^{-1}$

since after including (1,2)(1,3)(2,3) the relation is not symmetric so we have to make it symmetric by including necessary ordered pair.

GATE - 04

Q Consider the binary relation

$S = \{(x,y) \mid y = x+1$ and $x,y \in \{0,1,2,\ldots\}\}$

The reflexive transitive closure is

a) $\{(x,y) / y > x$ and $x,y \in \{0,1,2,\ldots\}\}$
b) $\{(x,y) \mid y \geq x$ and $x,y \in \{0,1,2,\ldots\}\}$
c) $\{(x,y) \mid y < x$ and $x,y \in \{0,1,2,\ldots\}\}$
d) $\{(x,y) \mid y \leq x$ and $x,y \in \{0,1,2,\ldots\}\}$

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$= \{(0,0)\ (1,1)\ (2,2) \ldots\ldots$

$(0,1)\ (1,2)\ (2,3) \quad (0,2)\ (1,3) \ldots\}$

$= \{(x,y) / y \geq x$, where $x,y \in \{0,1,2,\ldots\}\}$

B → option

GATE - 89

Q The transitive closure of the relation $\{(1,2)\ (2,3)\ (3,4)\ (5,4)\}$ on the set $A = \{1,2,3,4,5\}$ is

$R = \{(1,2)\ (2,3)\ (3,4),\ (5,4)\}$

$R^* = \{(1,2)(2,3)(3,4)(5,4)\ (1,3)\ (2,4)(1,4)\}$

Ans

Partial Order Relation

Let 'R' is a relation on set 'A' such that

i) Reflexive
ii) Antisymmetric
iii) Transitive ;

then [A ; R] is called POSET

A over R

Dn over divisibility

Divisibility sign

* Prove that [Dn ; ①] is a Partially ordered set [POSET]

[Note : Dn = set of all +ve divisors of 'n'
$D_8 = \{1, 2, 4, 8\}$, $D_6 = \{1, 2, 3, 6\}$]

xRx

* Reflexive : $\forall x \in D_n$, $x | x$

Here relation is Divisibility

* Antisymmetric :

$\forall x, y \in D_n$, if $x | y$ $\boxed{\text{and}}$ $y | x$

then $x = y$

* Transitive :

$\forall x, y, z \in D_n$

$2|8$

$2 \overline{) 8} (4$
$\underline{8}$
0

$8 = 2 \times 4$

if $x | y$ and $y | z$

$\Rightarrow$ $y = xp$ and $z = yq$

$x|y$

$y = x \times p$

here $z = yq$

$= xpq$ $(\because y = xp)$

$\Rightarrow$ $z = x(pq)$

$x | z$

$\therefore$ | is partial order Relation

$\therefore$ [Dn ; |] is POSET

* Prove that [A ; ≤] is a POSET, where A = set of all Real Numbers

i) Reflexive : $\forall x \in A$, xRx

(Here Relation is (≥≤))

$\forall x \in A$, $x \le x$

ii) Antisymmetric :

$\forall x, y \in A$, if $x \le y$ and $y \le x$ then

$x = y$

iii) Transitive :

$\forall x, y \in A$

if $x \le y$ and ~~$y \le x$~~ $y \le z$

then $x \le z$

$\therefore$ ≤ is partial order

$\therefore$ [A; ≤] is POSET

Dn over divisibility

↑ → Divisibility sign

* Prove that [Dn ; $\textcircled{|}$] is a Partially ordered set [POSET]

[Note : Dn = set of all +ve divisors of 'n'

$D_8 = \{1, 2, 4, 8\}$, $D_6 = \{1, 2, 3, 6\}$]

xRx

* Reflexive : $\forall x \in D_n$, $x | x$ [Here relation is Divisibility]

* Antisymmetric :

$\forall x, y \in D_n$, if $x | y$ $\boxed{\text{and}}$ $y | x$

then $x = y$

* Transitive :

$\forall x, y, z \in D_n$

if $x | y$ and $y | z$

$\Rightarrow$ $y = xp$ and $z = yq$

here $z = yq$

$= xpq$ ($\because y = xp$)

$\Rightarrow$ $z = x(pq)$

$x | z$

$2/8$

$2 \overline{)8} (4$
$\underline{8}$
0

$8 = 2 \times 4$

$x | y$

$y = x \times p$

$\therefore$ | is partial order Relation

$\therefore$ [Dn ; |] is POSET

* Prove that [A ; ≤] is a POSET, where A = set of all Real Numbers

i) Reflexive : $\forall x \in A$, ~~[illegible]~~ xRx

(Here Relation is ≤)

$\forall x \in A$, $x \leq x$

ii) Antisymmetric :

$\forall x, y \in A$, if $x \leq y$ and $y \leq x$ then

$x = y$

iii) Transitive :

$\forall x, y \in A$

if $x \leq y$ and ~~$y \leq x$~~ $y \leq z$

then $x \leq z$

$\therefore$ ≤ is partial order

$\therefore$ [A ; ≤] is POSET

* Prove that $[P(S) ; \subseteq]$ is POSET where $P(S)$ is power set of set 's'

$S = \{a, b\}$

$P(S) = \{\phi, \underbrace{\{a\}}_{M}, \underbrace{\{b\}}_{N}, \underbrace{\{a, b\}}_{O}\}$

i) Reflexive:

$\forall A \in P(S) \; ; \; A \subseteq A$

ii) Antisymmetric;

$\forall A, B \in P(S)$, if $A \subseteq B$ and $B \subseteq A$ then $A = B$

iii) Transitive

$\forall A, B, C \in P(S)$

if $A = B$ and $B = C$, then $A \subseteq C$

$\therefore \subseteq$ is Partial Order Relation

$\therefore [P(S) ; \subseteq]$ is POSET

POSET
$[D, |]$ $[D, \le]$ $[D, \subseteq]$

Hasse Diagram

Diagraph representation of a partial order Relation is called Hasse Diagram.

Ex:-1

Draw Hasse Diagram of POSET $[D_6, |]$

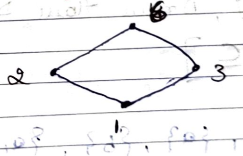
(Count no. of edges in Hasse Diagram of $[D_6, |]$ = ______)

$D_6 = \{1, 2, 3, 6\}$

(1,2) (2,6)
1 → POSET

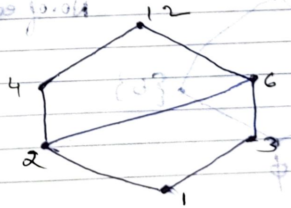
| = { ~~(1,1)~~ (1, 2) (1, 3) ~~(1, 6)~~ ~~(2, 2)~~
(2, 6) ~~(3, 3)~~, (3, 6) ~~(6, 6)~~ }

Remove self loops



ii) $[D_{12}, |]$

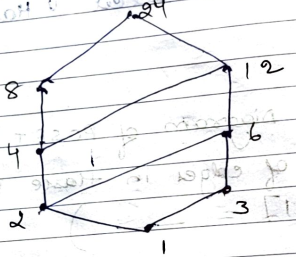
$D_{12} = \{1, 2, 3, 4, 6, 12\}$



$\therefore$ No. of edges = 7

iii) $[D_{24}, |]$

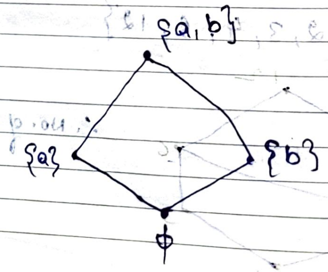
$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$



No. of edges = 10

Q) If $S = \{a, b\}$, Draw Hasse Diagram of $[P(S); \subseteq]$

$P(S) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$



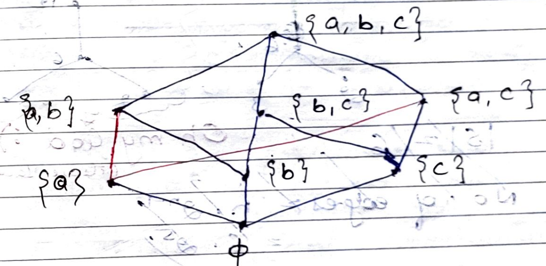
No. of edges = 4

$|S| = 2 = n$

$\therefore n \cdot 2^{n-1} = 2 \cdot 2^{2-1} = 4$

Q If $S = \{a, b, c\}$ Draw Hasse Diagram of $[P(S), \subseteq]$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}, \phi\}$



No. of edges = 12

2:04:30

$S = \{a, b, c\}$

$\therefore |S| = 3$

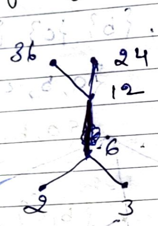
$\therefore n \cdot 2^{n-1} = 3 \cdot 2^{3-1} = 3 \cdot 2^2 = 12$

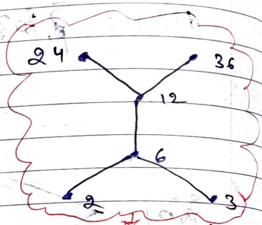
Short-cut Method

* If $|S| = n$ then no. of edges in Hasse diagram of $[P(S); \subseteq]$ $= n \cdot 2^{n-1}$

Q [{2,3,6,12,24,36}; |]

Find no. of edges in Hasse Diagram





Oh my God :)
(Please save me)

~~|S| = 6~~

~~No. of edges = $n \cdot 2^{n-1}$~~
~~$= 6 \cdot 2^5$~~
=

No. of edges = 5

[15:00]

Greatest lowerbound (GLB)(infimum)

Let 's' be a non empty subset of POSET 'P' an element a∈P is called lowerbound of 's' if aRx ∀x∈S

If 'a' is lowerbound of set 's' and 'b' is any other lowerbound of set 's' such that bRa

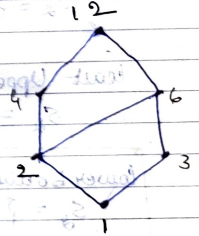
then 'a' is called G.L.B of 's'

Least Upperbound of a POSET (Supremum)

Let 'S' be a non empty subset of a POSET P an element a∈P is called an upperbound of S if xRa ∀x∈S.

If 'a' is an upperbound of 's' such that aRb for all upperbounds 'b' of 's' then the element 'a' is called least upperbound of 's'

	S={2,3}
Upperbound of S = {6,12}	2Ra 3Ra
least upperbound of S = {6}	
lowerbound of S = {1}	aR2 aR3
Greatest lower bound = {1}	

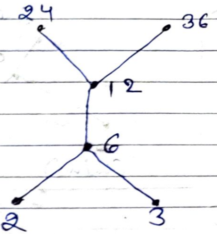


	S = {2, 3}
Upperbound of S = {6, 12, 24, 36}	2Ra 3Ra
least Upperbounds of S = {6}	
lowerbound of S = does not exist	aR2 aR3
Greatest lowerbound = does not exist	

	S_2 = {6, 12}
Upperbound of S_2 = {12, 24, 36}	6Ra 12Ra
least Upperbound of S_2 = {12}	
lowerbound of S_2 = {2, 3, 6}	aR6
Greatest lowerbound of S_2 = 6	aR12

S = {24, 36}

24Ra
36Ra



Upperbound of S
= does not exist

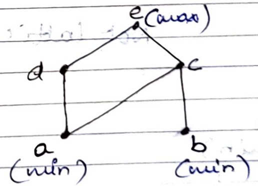
least upperbound of S = does not exist

lowerbound of S = {2, 3, 6, 12}

G.L.B of S = {12}

aR24
aR36

~~Max & At~~ Maximal & Minimal element

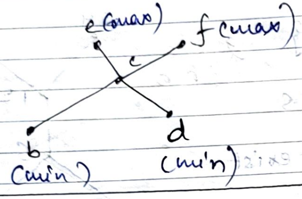


Not lattice

Maximal → No edge should starts.

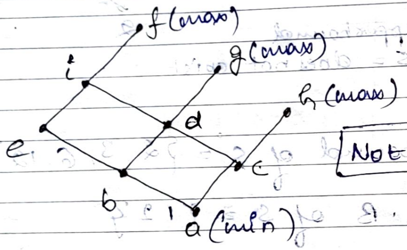
Minimal → No edge should terminate

②



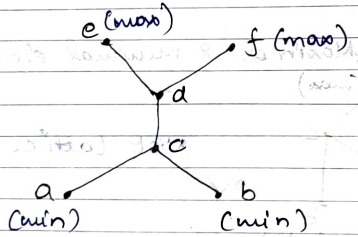
Not lattice

③



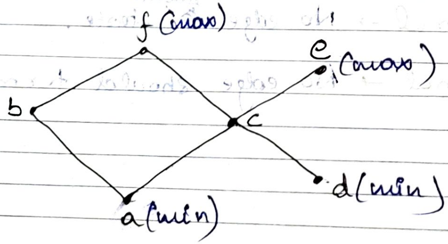
Not lattice

④



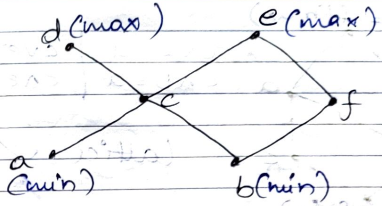
Not lattice

⑤



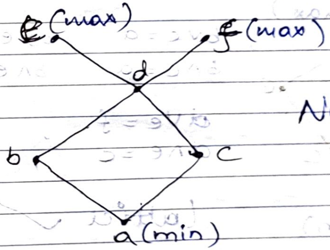
Not lattice

⑥



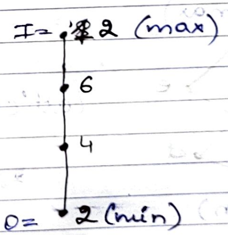
Not lattice

⑦



Not Lattice

⑧

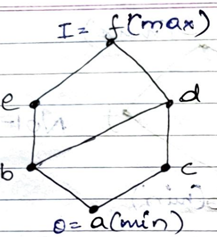


(Electric Pole)

Lattice ✓

[A = {2, 4, 6, 12} ; ≤]

⑨

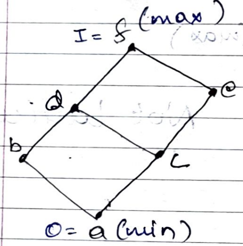


$c \vee b = d$	$c \vee e = f$
$c \wedge b = a$	$c \wedge e = a$

Lattice ✓

---

⑩



$b \vee c = d$ $b \vee e = f$

$b \wedge c = a$ $b \wedge e = a$

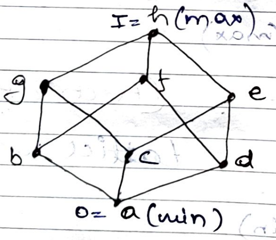
$d \vee e = f$

$d \wedge e = c$

Lattice ✓

---

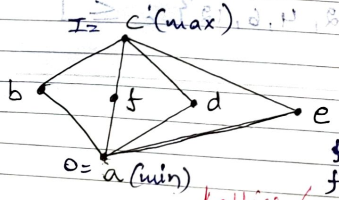
⑪



Lattice ✓

---

⑫

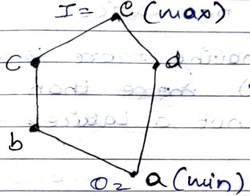


$b \vee f = c$	$b \vee d = c$	$b \vee e = c$
$b \wedge f = a$	$b \wedge d = a$	$b \wedge e = a$
$f \vee d = c$	$f \vee e = c$	$d \vee e = c$
$f \wedge d = a$	$f \wedge e = a$	$d \wedge e = a$

Lattice ✓

---

⑬



$b \vee d = e$

$b \wedge d = a$

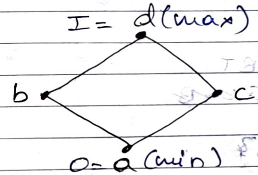
Lattice ✓

$c \vee d = e$

$c \wedge d = a$

---

⑭



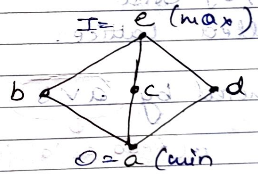
$b \vee c = d$

$b \wedge c = a$

Lattice ✓

---

⑮



$b \vee c = e$

$b \wedge c = a$

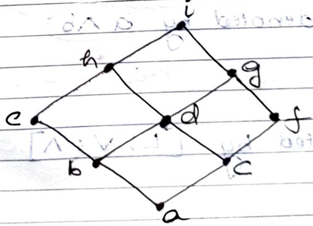
Lattice ✓

$b \vee d = e$

$b \wedge d = a$

---

⑯



→ Lattice ✓

Shortcut to ~~elem~~ eliminate option which are not Lattice

* If POSET is having more than one max (OR) more than one min then it is not a Lattice.

16|4|21 Lecture 6A

Lattice

Let P is a POSET

$\forall a, b \in P$, if ~~K·U·B~~

L·U·B of $\{a, b\}$

and G·L·B of $\{a, b\}$ exist then 'P' is called Lattice.

Note: LUB of $\{a,b\}$ denoted by $\boxed{a \vee b}$

(Read as 'a' join 'b')

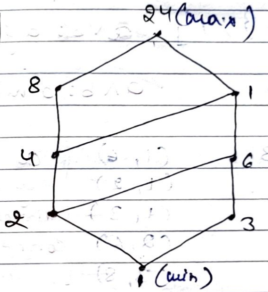
GLB of $\{a,b\}$ denoted by $\boxed{a \wedge b}$

(Read as 'a' meet 'b')

and lattice denoted by $[L; \vee; \wedge]$

Q Is $[D_{24}, |]$ Lattice ?

$[D_{24}, |]$



$2 \vee 3 = 6$
$2 \wedge 3 = 1$

$3 \vee 4 = 12$
$3 \wedge 4 = 1$

$3 \vee 8 = 24$
$3 \wedge 8 = 1$

$8 \vee 6 = 24$
$8 \wedge 6 = 2$

$4 \vee 6 = 12$
$4 \wedge 6 = 2$

$8 \vee 12 = 24$
$8 \wedge 12 = 4$

$\forall a, b \in [D_n, |]$

$a \vee b = LCM(a, b)$

$a \wedge b = G.C.D(a, b)$, $H.C.F(a, b)$

$\therefore [D_n, |]$ is always lattice

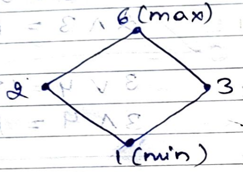
Q $[D_6, |]$

$[D_6 = \{1, 2, 3, 6\}; |]$

$\forall a, b \in P$

$a \vee b$: a join b

$a \wedge b$: a meet b

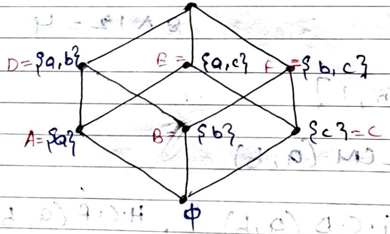


~~(1, 6)~~
~~(1, 2)~~
~~(1, 3)~~
~~(2, 6)~~
~~(3, 8)~~

as they are directly connected so there LUB and GLB exist

$2 \vee 3 = 6$
$2 \wedge 3 = 1$

ii) $S = \{a, b, c\}$



$\{a\} \vee \{b\} = \{a, b\}$
$\{a\} \wedge \{b\} = \phi$

$\{a\} \vee \{c\} = \{a, c\}$
$\{a\} \wedge \{c\} = \phi$

$\{a\} \vee \{b, c\} = \{a, b, c\}$
$\{a\} \wedge \{b, c\} = \phi$

$\forall A, B \in P(S)$

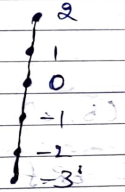
$A \vee B = A \cup B$
$A \wedge B = A \cap B$

$\therefore [P(S); \leq]$ is always lattice

iii) Let A = Set of all integers

Is $[A; \leq]$ lattice?

Ex: $[A = \{-3, -2, -1, 0, 1, 2\}; \leq]$



$-2 \vee 1 = 1$
$-2 \wedge 1 = -2$

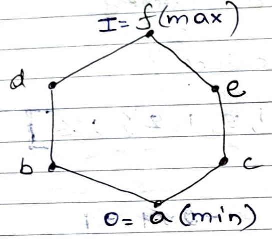
$\forall a, b \in A$

$a \vee b = \max(a, b)$
$a \wedge b = \min(a, b)$

YES, $[A; \leq]$ is lattice

1:07:00

Example where one max, one min exist but still it is not lattice



Upper bounds of (b, c)

bRe	bRd	bRf
cRe	cRd	cRf

∴ Upperbounds of $\{b, c\}$ $= \{e, d, f\}$

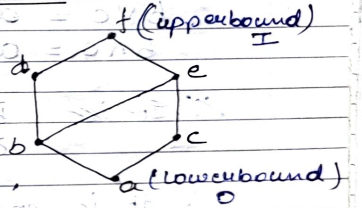
LUB of $\{b, c\}$ does not exist

∴ Not lattice.

* Every finite lattice is bounded

* $[A; \leq]$ where A is set of integers is unbounded lattice

Bounded lattice



let $[L, \vee, \wedge]$ is a lattice and $I \in L$, $O \in L$ $\forall x \in L$, if xRI and ORx then I is called upperbound, O is called lowerbound of lattice 'L' and 'L' is said to be bounded lattice.

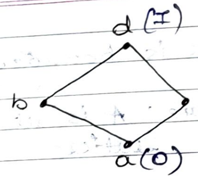
Complemented lattice

Let $[L, \vee, \wedge]$ is a bounded lattice with upper bound 'I' and lowerbound 'O'.

for $x \in L$, such that, for $y \in L$ $x \vee y = I$ and $x \wedge y = O$ then 'y' is called complement of x.

($\bar{a} \to$ a complement)

Page No.
Date



$\bar{a} = d$

$\bar{d} = a$

$b \vee c = d(I)$

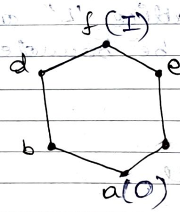
$b \wedge c = a(O)$

∴ ~~It is~~ $\bar{b} = c$

$\bar{c} = b$

∴ It is complemented lattice

1:39



$\bar{a} = f$, $\bar{f} = a$

$b \vee e = f(I)$

$b \wedge e = a(O)$

∴ $\bar{b} = e$, $\bar{e} = b$

$c \vee \_\_ = f(I)$
$c \wedge \_\_ = a(O)$ } $\bar{c}$ does not exist

Similarly $\bar{d}$ does not exist

∴ It is not a complemented lattice.

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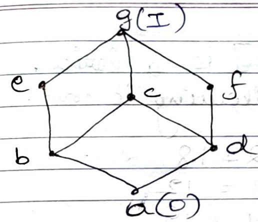


$\bar{a} = g$, $\bar{g} = a$

$\bar{b} = f$, $\bar{f} = b$

$\bar{d} = e$, $\bar{e} = d$

$\bar{e} = f$, $\bar{f} = e$

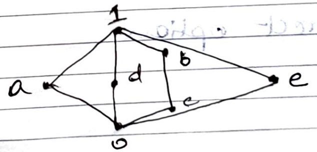
$\bar{c}$ does not exist

∴ not complemented lattice

Note:- An element can have more than one complement

Note:- If every element of 'L' has atleast one complement then it is called complemented lattice.

Q The complement(s) of the element 'a' in the lattice shown in fig is (are)



$\bar{a} = d$, $\bar{a} = b$, $\bar{a} = c$, $\bar{a} = e$ Aw

$\bar{d} = a$, $\bar{d} = b$, $\bar{d} = c$, $\bar{d} = e$

$\bar{0} = 1$, $\bar{1} = 0$

$\bar{b} = d$, $\bar{b} = a$, $\bar{b} = e$

$\bar{c} = a$, $\bar{c} = d$, $\bar{c} = e$

Q The poset $[D_{36} ; |]$ is a lattice, which of the following is true.

a) Complement of $2 = 18$
b) " " $3 = 12$
c) " " $4 = 9$
d) " " $6 = 1$

$I = 36, \quad O = 1$

We know that $\forall a, b \in D_n$,

$a \vee b = LCM(a,b)$
$a \wedge b = HCF(a,b)$ or $G.C.D(a,b)$

$2 \vee 18 \neq 36\ (I)$ | $3 \vee 12 \neq 36\ (I)$
$2 \wedge 18 \neq 1\ (O)$ | $3 \wedge 12 \neq 1\ (O)$

$4 \vee 9 = 36$
$4 \wedge 9 = 1$ ✓

c → Correct option

Distributive lattice

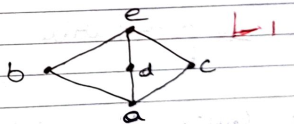
Let $[L, \vee, \wedge]$ is a lattice
$\forall a, b, c \in L$ if

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and

ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Then L is called Distributive lattice

Q Which of the following are Distributive lattice

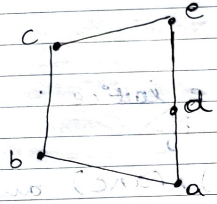


$\Rightarrow b \vee (d \wedge c) = (b \vee d) \wedge (b \vee c)$

$b \vee a = e \wedge e$

$b = e$ False

(Kite) this (4) structure is not distributive



L2 ~~strict~~ lattice

$b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$

$b \vee a = c \wedge e$

$b = c$ (FALSE)

L2 lattice not ~~dis~~ distributive lattice.

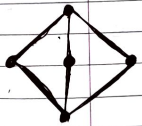
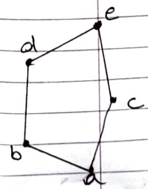
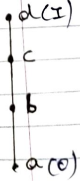
NOTE:- In a Distributive lattice, if complement of an element exist, then that complement is unique.

✱ Boolean Algebra

If (L, ∨, ∧) is a lattice and if 'L' is

i) Complemented and
ii) Distributive ~~than~~

then, 'L' is Boolean Algebra.

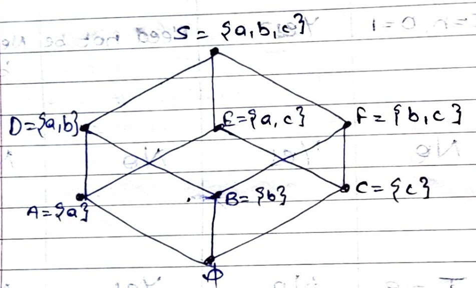
Lattice	Bounded Lattice	Distributive lattice	Complemented Lattice	Boolean Algebra
[P(S); ⊆]	I = S; 0 = ϕ	Yes	Yes ∀ A ∈ P(S); $\bar{A} = S - A$	Yes
[Dn; |]	I = n, 0 = 1	Yes	Need not be	Need not be
[A; ≤] where A = integer	No	Yes	No	No
	I = e 0 = a	No	Yes	No
	I = e 0 = a	No	Yes	No.
	I = d 0 = a	Yes	No $\bar{a} = d$ $\bar{d} = a$ Remaing elements are not having complement	No

Note; $[P(S); \subseteq]$ is complemented lattice;

~~$S = \{a, b, c\}$~~ $\forall A \in P(S)$

~~$[P(S); \subseteq]$~~ $\bar{A} = S - A$

$S = \{a, b, c\}$, $[P(S); \subseteq]$



$\{a\} \vee \{b, c\} = \{a, b, c\}$ I

$\{a\} \wedge \{b, c\} = \phi$ 0

$\therefore \{\bar{a}\} = \{b, c\}$

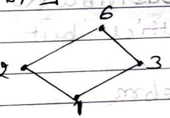
$\bar{A} = S - A$

Totally Ordered Set [TOSET]

Let 'S' be non empty set and 'R' is a relation defined on 'S'.

$\forall a, b \in S$, either aRb (or) bRa then 'R' is called Totally ordered Relation and $[A; R]$ is called TOSET (chain)

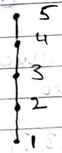
$[D_6, |]$



here for $2, 3 \in D_6$, neither $2R3$ nor $3R2$

$\therefore [D_6, |]$ not a TOSET

* $[A = \{1, 2, 3, 4, 5\}, \leq]$ is a TOSET

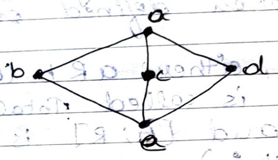


Note:- Every TOSET is Distributive.

GATE

Q The following is the Hasse diagram of the poset [{a,b,c,d,e}, ≤]

The poset is



a) Not a lattice

b) a lattice but not distributive lattice

c) a distributive lattice but not a Boolean Algebra.

d) a Boolean Algebra

Since it is a kite structure so it is not distributive but it is a lattice.

b → correct option

GATE

Q) The inclusion of which of the following sets into

$S = \{\{1,2\}, \{1,2,3\}, \{1,3,5\}, \{1,2,4\}\{1,2,3,4,5\}\}$

is necessary and sufficient to make S a complete lattice under the partial order defined by set containment?

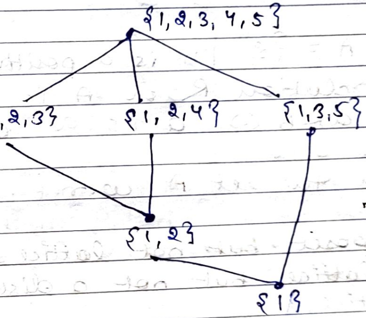
$\subseteq$

a) {1}

b) {1}, {2,3}

c) {1}, {1,3}

d) {1}, {1,3}, {1,2,3,4}, {1,2,3,5}



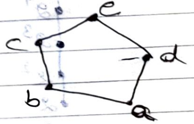
{1} → necessary & sufficient

A → correct option

Q Consider the set X = {a,b,c,d,e} under the partial ordering.

R = {(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)}

The Hasse diagram of the partial order (X,R) is shown



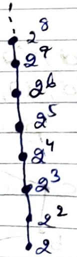
The minimum number of ordered pairs that needs to be added to R to make (X,R) a lattice is ____.

Q) Since it is already a lattice.
So no need to add any other ordered pair.
∴ 0

Q Let $A = \{2^n \mid n$ is a positive integer$\}$.
A relation R on A is defined by aRb ⟺ a is a divisor of b.
Then the set A w.r.t. R is

a) a poset but not lattice
b) a lattice but not a distributive lattice.
c) a distributive lattice but not a bounded lattice
d) not a POSET.

$[A = \{2^1, 2^2, 2^3, 2^4, ---\}; |]$



It is TOSET
Every POSET is Distributive

& It is not bounded,

H.W

Q Let $A = \{1, 2, 3, ---, 10\}$. A relation R on A is defined by
"aRb" iff a is a divisor of b"
Number of edges in the Hasse diagram of the POSET [A;R] is ______.

11 → edges

Q Let R be a partial ordering relation on the set $A = \{1, 2, 3, 4\}$ such that $R \cup R^{-1} = A \times A$.
Consider the following statements.

S_1: The poset [A;R] is a distributive lattice.
S_2: The poset [A;R] is a complemented lattice.

Which of the following is TRUE.

a) S_1 is TRUE and S_2 is false.
b) S_1 is FALSE and S_2 is TRUE.
c) Both S_1 & S_2 are TRUE
d) Both S_1 & S_2 are FALSE.

a) option

In the above problem R must be Total Order Relation then only $R \cup R^{-1} = A \times A$ and we know that every TOSET is distributive and in Total Order sets only upper bound & lower bound having complements so it is not a complemented lattice.
1st ~~option~~

Sub lattice :-

Suppose M is a non empty subset of a lattice 'L' we say 'M' is a sublattice of 'L' if

i) M is a lattice
ii) LUB (GLB) of any two element a, b in M is same as LUB (GLB) of a and B in 'L'.

Ex:- for the lattice $[A; \leq]$, where $A = \{1, 2, 3, \ldots, 10\}$ let $M = \{1, 2, 3\}$ then $[M; \leq]$ is a sublattice.

Step i) :-

$M \subseteq A$ and $[M; \leq]$ is a lattice

Step (ii) :-

LUB {2, 3} in M = max(2, 3) = 3

GLB {2, 3} in M = 2

LUB {2, 3} in A = max (2, 3) = 3

GLB {2, 3} in A = 2

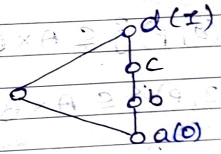
Ex-2 Let the given lattice is $[A; |]$ where

$A = \{1, 2, 3, 4, 6, 12\}$

now $B = \{1, 2, 3, 6\}$ is a sublattice

i.e $[B; |]$ is sublattice of $[A; |]$

Answer of prev. problem

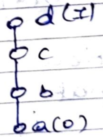


$\bar{a} = d$, $\bar{d} = a$
$\bar{b} = e$, $\bar{e} = b$
$\bar{c} = e$, $\bar{e} = c$

→ Complemented lattice

S is a complemented lattice

M =



→ Not complemented lattice.

18/4/21 Lecture 7A

Function

Let $A = \{1, 2, 3\}$ $B = \{p, q, r, s\}$

$A \times B = \{(1,p)(1,q)(1,r)(1,s),(2,p)(2,q)$
$(2,r)(2,s)(3,p)(3,q)(3,r)(3,s)\}$

$\therefore$ No. of Relations from A to B
$= |P(A \times B)| = 2^{12}$

$R_1 = \{(1,p)(1,q)(2,p)\} \subseteq A \times B$

$R_2 = \{(1,p)(2,p)(3,p)\} \subseteq A \times B$

$R_3 = \{(1,p)(2,q)(3,r)\} \subseteq A \times B$

$R_4 = \{(1,p)(2,q)(3,r)(2,s)\} \subseteq A \times B$

Let 'f' is a Relation from A to B such that every element of 'A' is uniquely mapped with some element of B, then 'f' is called function from A to B.

13:30

Range = Codomain

$A = \{1, 2, 3\}$, $B = \{p, q, r\}$, $C = \{a, b\}$, $D = \{m, n, o, p\}$

One-One
Onto

		One-One	Onto	Bijective
$f: A \to B$ $f = \{(1,p),(2,q)(3,r)\}$	Domain = A $= \{1, 2, 3\}$ Codomain = B $= \{p, q, r\}$ Range of $f = \{p, q, r\}$	✓	✓	✓
$g: A \to C$ $g = \{(1,a)(2,b)(3,b)\}$	Domain of $A = \{1,2,3\}$ Codomain $= C = \{a, b\}$ Range of $g = \{a, b\}$	✗	✓	✗
$h: A \to D$ $h = \{(1,m),(2,n)(3,o)\}$	Domain $= A = \{1,2,3\}$ Codomain = D $= \{m, n, o, p\}$ Range of $h =$ $\{m, n, o\}$	✓	✗	✗
$K: A \to B$ $K = \{(1,p)(2,p)(3,q)\}$	Domain $= A = \{1,2,3\}$ Codomain = B $= \{r, q, p\}$ Range of $K =$ $= \{p, q\}$	✗	✗	✗
$L: A \to C$ $L = \{(1,b),(2,b)(3,a)\}$	Domain = A $= \{1, 2, 3\}$ Codomain = C $= \{a, b\}$ Range of $L = \{b, a\}$	✗	✓	✗

→ J: A → D

$J = \{(1,m)(2,n)(3,o),(1,n)\}$

(Not a function)

→ A → D

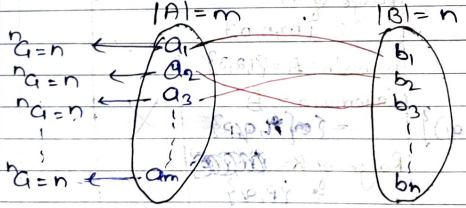
$K = \{(1,m)(2,n)\}$

(Not function)

Note: ① If $|A| = m$, $|B| = n$ then

a) No. of function from A to B $= |B|^{|A|}$

$|A| = m$ $|B| = n$



$$= \underbrace{n \cdot n \cdot n \cdots n}_{m \text{ times}}$$

$$= n^m$$

$$= |B|^{|A|}$$

GATE-98

Q Suppose X and Y are sets and |X| and |Y| are their respective cardinalities. It is given that there are exactly 97 functions from X to Y. From this, one can conclude that.

a) $|X| = 1$, $|Y| = 97$
b) $|X| = 97$, $|Y| = 1$
c) $|X| = 97$, $|Y| = 97$
d) None of the above

No. of function $= |Y|^{|X|} = 97$

$\therefore Y = 97$, $X = 1$

A → option

Q The no. of functions from an m element set to an n element set is

a) m^n b) $m+n$ c) n^m d) $m*n$

$m \to n$

C → option $(n^m) =$ No. of f^n's

GATE-98

Q A function $f: N^+ \to N^+$ defined on the set of positive integers N^+, satisfies the following properties:

H.W

$f(n) = f(n/2)$ if n is even

$f(n) = f(n+5)$ if n is odd

Let $R = \{i \mid \exists j : f(j) = i\}$ be the set of distinct values that f takes. The maximum possible size of R is

GATE = 06

Q Let X, Y, Z be ~~too~~ sets of sizes x, y, z resp. Let $W = X \underset{\downarrow \times}{*} Y$ and E be the set of all subset of W. The no. of functions from Z to E is

a) 2^{xy} b) 2×2^{xy}

c) $2^{2^{x+y}}$ d) 2^{xyz}

$|W| = |X \times Y| = xy$

$|E| = |\text{powerset of } W| = 2^{xy}$

No. of functions from Z to E $= |E|^{|Z|} = (2^{xy})^{z}$

$= 2^{xyz}$

D → Correct Option

GATE -14- Set 1

Q Let S denote the set of all functions $f : \{0,1\}^4 \to \{0,1\}$. Denote by N the no. of functions from S to the set $\{0,1\}$. The value of $\log_2 \log_2 N$ is ____

A $\{0,1\}^2 = \{0,1\} \times \{0,1\} = \{(0,0)(0,1)(1,0)(1,1)\}$

$|\{0,1\}^4| = 2^4 = 16$

$f : \{0,1\}^4 \to \{0,1\}$

$\therefore |S| = 2^{2^4} = 2^{16}$

N = No. of function from 'S' to $\{0,1\}$

$= 2^{|S|}$

$= 2^{2^{16}}$

$\log_2 \log_2 N = \log_2 \log_2 2^{2^{16}}$

$= \log_2 2^{16} = \cancel{16}$

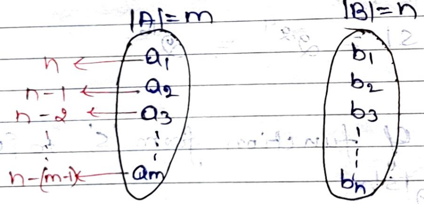
$= 16$

~~Q Let f : A → B~~

Q Workbook Q: 28

Note: ① The necessary condition to define one-one from set A to set B is $|A| \le |B|$

② If $|A| = m$, $|B| = n$, where $m \le n$, then how many no. of one-one functions are possible from set A to set B.



$= n(n-1)(n-2)\ldots\ldots(n-(m-1))$

$= {}^{n}P_{m}$

$= {}^{|B|}P_{|A|}$

$\because {}^{n}P_{m} = \frac{n!}{(n-m)!}$

$= \frac{n(n-1)\ldots(n-(m-1))\cancel{(n-m)!}}{\cancel{(n-m)!}}$

(GATE = 15 - set 2)

Q Let X and Y denote the sets containing 2 & 20 distinct objects resp. and F denote the set of all possible f^n's defined from X to Y. Let f be randomly choosen from F. The probability of f being one to one is ____

$|X| = 2 \quad |Y| = 20$

$\therefore$ Total no. of functions $= |Y|^{|X|} = 20^{2}$

$\therefore$ No. of one-one

$= {}^{|Y|}P_{|X|}$

$= {}^{20}P_{2} = \frac{20!}{18!} = 20 \times 19$

P(f being one-one)

$= \frac{\text{Favorable (i.e no. of one-one } f^n\text{'s)}}{\text{Total no. of functions}}$

$= \frac{20 \times 19}{20 \times 20}$

$= \frac{19}{20}$

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Note :- ① The necessary condition to define onto from A to B is $|A| \geq |B|$

II) If $|A| = m$, $|B| = n$ $(m \geq n)$ then how many no. of onto functions are possible from A to B

$$= n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \ldots + (-1)^{n-1}\,{}^nC_{n-1}(1)^m$$

Proof

If $|A| = m$, $|B| = n$

No. of onto functions

$$= \left[\text{Total no. of functions}\right] - \left[\text{Total no. of non-onto functions}\right]$$

$$= n^m - \left[{}^nC_1(n-1)^m - {}^nC_2(n-2)^m + {}^nC_3(n-3)^m - \ldots \pm (-1)^{n-1}\,{}^nC_{n-1}(1)^m\right]$$

$A = \{1, 2, 3\}$, $B = \{p, q, r\}$

Onto ; Range = Codomain

Non Onto: Range ≠ Codomain

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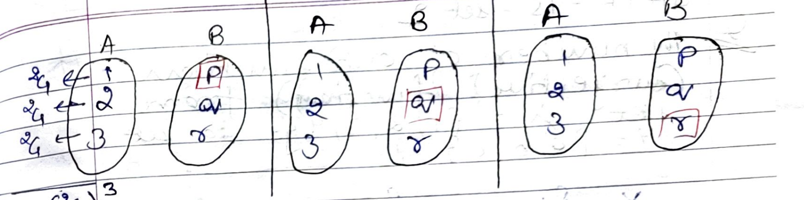
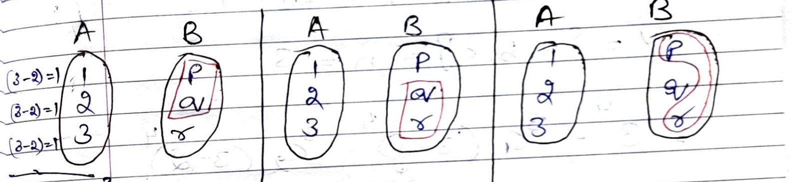


$3 \cdot ({}^2C_1)^3$



$3 \cdot (1)^3$

$|A| = m$ $\qquad$ $|B| = n$

Total no. of non-Onto functions :-

$$= {}^nC_1(n-1)^m - {}^nC_2(n-2)^m + {}^nC_3(n-3)^m - \ldots - (-1)^{n-1}\,{}^nC_{n-1}(1)^m$$

The placing of add^n & subtraction sign depends on the principle of inclusion & exclusion

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

GATE - 15 - set 2

Q The number of onto function (surjective function) from set $X = \{1, 2, 3, 4\}$ to set $Y = \{a, b, c\}$ is ______.

$X = \{1, 2, 3, 4\} = m = 4$

$Y = \{a, b, c\} = n = 3$

$= 3^4 - {}^3C_1 (3-\underset{\text{elements blocked}}{1})^4 + {}^3C_2 (3-\underset{\text{elements blocked}}{2})^4$

$= 81 - 48 + 3$

$= 36$

Don't rote the formula try to solve Questions by concept and learn the formula by making sentence, :)

19/4/21

Lecture 7B

Inverse of a function:-

Let 'f' is a function from set A to set B if the inverse relation of f is also a function from set B to set A then it is called ~~inf~~ inverse of a function 'f'.

* $A = \{1, 2, 3\}$, $B = \{p, q, r, s\}$

$f := \{(1,p)\,(2,q)\,(3,p)\}$ is a function from A to B

$f^{-1} = \{(p,1)\,(q,2)\,(p,3)\}$ is not a function from B to A

$\therefore f^{-1}$ does not exist

$g: \{(1,p)\,(2,q)\,(3,r)\}$ function from A to B

$g^{-1} := \{(p,1)(q,2)(r,3)\}$ is not a function from B to A

$|A| \geq |B|$ $|A| \leq |B|$

Note:

① $f: A \to B$ is a Bijective function iff f is invertible

② Necessary condition for defining inverse of a function is $|A| = |B|$

Ex:- let $A = \{1,2,3\}$, $B = \{p,q,r\}$

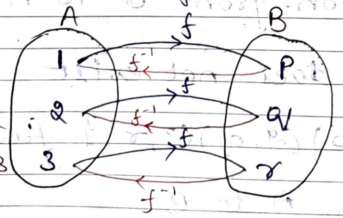
$f = \{(1,p)(2,q)(3,r)\}$ is a function from A to B

$f^{-1} = \{(p,1)(q,2)(r,3)\}$ is also function from B to A

$f^{-1}(p)=1$

$f^{-1}(q)=2$

$f^{-1}(r)=3$



$f(1)=p$

$f(2)=q$

$f(3)=r$

* If $|A| = |B| = n$, then total no. of Bijective from A to B = n!

$|A| = m$, $|B| = n$

$$^nP_m = {}^nP_n = \frac{n!}{(n-n)!} = n!$$

GATE-12

Q How many onto (or surjective) functions are there from an n-element ($n \geq 2$) set to a 2 element set?

a) 2^n b) $2^n - 1$

c) $2^n - 2$ d) $2(2^n - 2)$

$|A| = n$, $|B| = 2$

$= 2^n - {}^2C_1 (2-1)^n$

$= 2^n - 2$ (c-option)

GATE→96

Q Let R denote the set of real numbers let $f: R\times R \to R\times R$ be a bijective function defined by $f(x,y) = (x+y, x-y)$. The inverse function of f is given by

a) $f^{-1}(x,y) = \left(\frac{1}{x+y}, \frac{1}{x-y}\right)$

b) $f^{-1}(x,y) = (x-y, x+y)$

c) $f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$

d) $f^{-1}(x,y) = (2(x-y), 2(x+y))$

$f(\overset{3}{x},\overset{2}{y}) = f(\overset{5}{x+y}, \overset{1}{x-y})$

$f^{-1}(\overset{5}{x+y}, \overset{1}{x-y}) = (\overset{3}{x}, \overset{2}{y})$

$f^{-1}(x,y) = ?$

$\therefore f^{-1}(5,1) = \left(\frac{5+1}{2}, \frac{5-1}{2}\right)$

Put (5,1) in option and whoever match

$f^{-1}(5,1) = (3,2)$

that will be the correct option

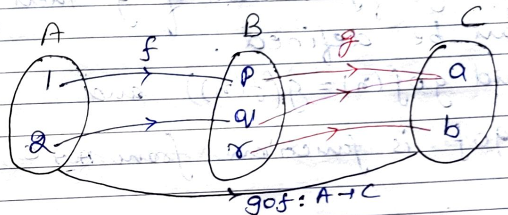
* $f^{-1}(x+y, x-y) = (x,y)$

$x+y = a$, $x-y = b$

$f^{-1}(a,b) = \left(\frac{a+b}{2}, \frac{a-b}{2}\right)$

∴ Option C is correct

Composition of a function:-



$f: A \to B$ such that $f = \{(1,p)\ (2,q)\}$

$g: B \to C$ such that $g = \{(p,a), (q,a)\ (r,b)\}$

$gof: A \to C$ defined as

$gof = \{(1,a)\ (2,a)\}$

* $A = \{1,2\}$, $B = \{p,q,r\}$, $C = \{a,b\}$

$gof: A \to C$

$gof(x) = g(f(x))$

$gof(1) = g(f(1)) = g(p) = a$

$gof(2) = g(f(2)) = g(q) = a$

Note :-
If Range of $f \subseteq$ Domain of 'g' then gof can be defined.

and $gof(x) = g(f(x))$ and

gof is function from $A \to C$

Note :-
In the above example fog does not exist because ~~Range~~ .

Range of g $\not\subseteq$ Domain of f

Q $A = \{m, n, p\}$ $B = \{l, o, r\}$ $C = \{1, 2, 3\}$

$K: B \to C$ such that

$K = \{(l,2), (o,3) (r,2)\}$

$T: A \to B$ such that

$T = \{(m,l) (n,r) (p,o)\}$

Is KoT exist ? (Yes/no) Yes

Is ToK exist ? (Yes/No) No

KoT : $A \to C$ exist

Q $A = \{1, 2, 3\}$, $B = \{4,5,6\}$, $C = \{l, m, n\}$

$f: A \to C$: $f: \{(1, l), (2, m) (3, n)\}$

$g: C \to A$: $g: \{(l, 1) (m, 2) (n, 3)\}$

Is fog & gof exist ? (Yes/No).

* gof : $A \to A$:

$gof = \{(1,1) (2,2) (3,3)\}$

* fog : $C \to C$

$fog = \{(l,l) (m,m) (n,n)\}$

Is gof exist ? (Yes/No) Yes ✓

Is fog exist ? (Yes/No) Yes ✓

fog & gof exist.

Properties :-

Let $f: A \to B$, $g: B \to C$ are two functions;

$gof : A \to C$

i) If both f and g are one-one then gof is one-one.

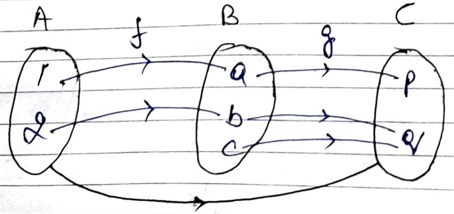
ii) If both f and g are onto then gof is onto.

iii) If both f and g are bijective then gof is also bijective.

iv) If gof is one-one then 'f' is one-one.

v) If gof is onto then 'g' is onto.

vi) If gof is bijective then f is one-one and g is onto.



$f: A \to B$; $f = \{(1,a)\ (2,b)\}$

$g: B \to C$; $g = \{(a,p)\ (b,q)\ (c,q)\}$

$gof : A \to C$: $gof = \{(1,p)\ (2,q)\}$

Range of gof $= \{p,q\}$

Codomain of gof $= \{p,q\}$

∴ Range = Codomain

∴ gof is onto

and 'g' is onto but 'f' is not onto

i) hog: one-one

↓

g is one-one

ii) hog: onto

↓

h is onto

GATE-05

Q Let 'f' be a function from a ~~set~~ set A to a set B, 'g' be a function from B to C and 'h' be a function from A to C, such that

$h(a) = g(f(a)) \ \forall \ a \in A$.

which of the following statements is always true for all such functions ~~f and~~ f and g?

a) g is onto ⇒ h is onto

b) h is onto ⇒ f is onto

c) h is onto ⇒ g is onto

d) h is onto ⇒ f and g are onto.

Sol $f : A \to B$

$g : B \to C$

$h(a) = g(f(a))$

$= gof(a)$ is one-one

then 'f' is one-one

gof(a) is onto then g is onto.

C → option correct

identity, constant f^n

Proof of one-one, onto

— × — × FUNCTION × — × — × — × —

Lecture - 8A

20/4/21

Groups

i) Group
ii) Subgroup
iii) Cyclic group

	Closure	Associative	Identity	Inverse	Commutative
1) Quasigroup	✓				
2) Semigroup	✓	✓			
3) Monoid	✓	✓	✓		
4) Group	✓	✓	✓	✓	
5) Abelian Group	✓	✓	✓	✓	✓

Group

1) Definitions

- ~~Algebraic~~ Algebraic Structure
- Closure Property
- Associative Property
- Identity Property • Inverse Property
- Commutative Property
- Order of a group **
- Order of an element
- Examples for finite and infinite group
- Addition modulo *****
- Multiplication modulo *****
- Properties of a Group **

Subgroup

i) Definitions
ii) Examples of subgroup
iii) Legrange's Theorm
iv) Properties of a subgroup

Cyclic Group

i) Definition, Examples of cyclic group
ii) Generator
iii) Properties of cyclic group.

#

$\mathbb{N}$ = Set of all Natural Numbers

$= \{1, 2, 3, ---\}$

$\mathbb{Z}$ = Set of all integers

$= \{--- -3, -2, -1, 0, 1, 2, 3\}$

$\mathbb{W}$ = Whole Numbers $= \{0, 1, 2, 3, ---\}$

$\mathbb{Q}$ = Set of all Rational Numbers

$= \{\frac{p}{q} / q \neq 0, \text{ and } (p, q) = 1\}$

$\mathbb{Q}'$ = Irrational Numbers $= \{\sqrt{2}, \sqrt{3}, \sqrt{5}, ---\}$

$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$

$\mathbb{Q}_0 = \mathbb{Q} - \{0\}$

$\mathbb{R}_0 = \mathbb{R} - \{0\}$

$\mathbb{C} = \{a + ib / a, b \in \mathbb{R}\}$

$\mathbb{C}_0 = \mathbb{C} - \{0\}$

Quasigroup

Binary Operation

Let 'G' be a non-empty set $\forall\ a,b \in G$, if ~~$a*b$~~ $a*b \in G$ then '$*$' is called Binary Operation on 'G' and G is said to be satisfying closure property with respect to '$*$' and the structure $(G,*)$ is called Quasigroup.

Ex:1

Let $\mathbb{N}$ is natural number set $\forall\ a,b \in \mathbb{N}$, we know that $a+b \in \mathbb{N}$

$\therefore$ '$+$' is Binary operation on $\mathbb{N}$

$\mathbb{N}$ is satisfying closure property

$(\mathbb{N},+)$ is called Quasigroup

Ex:2

$\forall\ a,b \in \mathbb{N}$, $a-b \in \mathbb{N}$ need not be True

e.g $3,5 \in \mathbb{N}$, $3-5=-2 \notin \mathbb{N}$

$\therefore$ N is not satisfying closure property w.r.t '$-$'

$\therefore$ $(\mathbb{N},-)$ is not Quasigroup

Ex.3 $\forall\ a,b \in Q$, $a*b = \frac{ab}{3}$

Is $(Q,*)$ Quasigroup?

Sol

$\forall\ a,b \in Q$, $ab \in Q$

$\Rightarrow \frac{ab}{3} \in Q$

$\Rightarrow a*b \in Q$

$\therefore$ $(Q,*)$ is Quasigroup

Ex4 $\forall\ a,b \in \mathbb{N}$, $a*b = a^b$.

Is $(\mathbb{N},*)$ Quasigroup?

$\forall\ a,b \in \mathbb{N}$, $a^b \in \mathbb{N}$

$\Rightarrow a*b \in \mathbb{N}$

$\therefore$ $(\mathbb{N},*)$ is Quasigroup.

Ex.5

$\forall\, a, b \in Q,\ a * b = a^b$. Is $(Q,*)$ Quasigroup

$\forall\, a, b \in Q,\ a^b \in Q$ .need not be TRUE

$$\left.\begin{array}{l} a = 2 \in Q \\ b = \frac{1}{2} \in Q \end{array}\right\} \quad a^b = 2^{1/2} \notin Q$$

$\therefore (Q,*)$ not Quasigroup

Ex.6

Let V = set of all vectors.

$= \{a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \,/\, \hat{i}, \hat{j}, \hat{k}$ are called unit vectors $a, b, c \in \mathbb{R}\}$

i) $(V,+)$ Quasigroup?

$\forall\, \hat{a}, \hat{b} \in V$, we know that

$\vec{a} + \vec{b} \in V$

$\therefore (V,+)$ is Quasigroup

ii) $(V,\cdot)$ Quasigroup? (here $\cdot$ is dot product)

$\forall\, \vec{a}, \vec{b} \in V,\ \vec{a}\cdot\vec{b} \notin V$

Let $\vec{a} = 2\hat{i} + 3\hat{j}$, $\vec{b} = 3\hat{i} + 2\hat{j}$

$\therefore \vec{a}\cdot\vec{b} = 12 \notin V$

$\therefore (V,\cdot)$ not Quasigroup

iii) $(V,\times)$ Quasigroup (here $\times$ is cross product)

$\forall\, \vec{a}, \vec{b} \in V,\ \vec{a}\times\vec{b} \in V$

$\therefore (V,\times)$ is Quasigroup.

Associative Property :-

Let 'G' be a non empty set and '*' is binary operation on 'G' such that

$\forall a,b,c \in G$,

if $a*(b*c) = (a*b)*c$

then 'G' is said to be satisfying Associative property w.r.t '*'

and $(G,*)$ is called Semigroup

i.e if 'G' satisfies

i) Closure and
ii) Associative

then $(G,*)$ is semigroup.

Ex.1 $\forall a,b \in Q$, $a*b = \frac{ab}{3}$

Is $(Q,*)$ semigroup?

$\forall a,b,c \in Q$, we have to prove that

$$a*(b*C) = (a*b)*c$$

we already proved it is satisfying closure property.

$$a* \frac{bc}{3} = \frac{ab}{3} * c$$

$$\frac{a \cdot \frac{bc}{3}}{3} = \frac{\frac{ab}{3} \cdot c}{3}$$

$$\frac{abc}{9} = \frac{abc}{9}$$

L.H.S = R.H.S

$\therefore (Q,*)$ is semigroup

Ex.2 $\forall a,b \in Q$, $a*b = a^2+b^2$

Is $(Q,*)$ Semigroup?

Closure :-

$\forall a,b \in Q$, $a^2 \in Q$, $b^2 \in Q$

$\Rightarrow a^2+b^2 \in Q$

$\Rightarrow a*b \in Q$

$\therefore (Q,*)$ is Quasigroup

Associative :-

$\forall a,b,c \in Q$

$$a*(b*c) = (a*b)*c$$

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L.H.S R.H.S

$\Rightarrow a * (b^2 + c^2)$ $\Rightarrow (a^2+b^2) * c$

$\Rightarrow a^2 + (b^2+c^2)^2$ $\Rightarrow (a^2+b^2)^2 + c^2$

$a^2 + (b^2+c^2)^2 = (a^2+b^2)^2 + c^2$

need not be TRUE

→ $\therefore (Q, *)$ is not semigroup

Ex.3 $\forall a, b \in N$, $a * b = a^b$,

Is $(N, *)$ Semigroup ?

Closure :-

$\forall a, b \in N$, $a^b \in N$

$\therefore (N, *)$ is Quasigroup

Associative :-

$\forall a, b, c \in N$

$a*(b*c) = (a*b)*c$

$a * b^c = a^b * c$

$a^{b^c} = (a^b)^c$

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$a=2, b=3, c=4$

$2^{12} \neq 2^{81}$

$\therefore a^{b^c} = a^{bc}$ need not be TRUE

$\therefore (N, *)$ is not Semigroup.

Identity Property :-

Let 'G' be a non empty set with '$*$' is Binary Operation on 'G'.

$\forall a \in G$, $\exists e \in G$ such that

$a * e = e * a = a$

then 'e' is called identity in 'G'.

If 'G' satisfies
i) Closure
ii) Associative and
iii) Identity

then $(G, *)$ is called monoid.

Ex1 $\forall a, b \in Q$, $a * b = \frac{ab}{3}$;

Is $(Q, *)$ monoid ?

We already proved $(Q, *)$ is Semigroup.

let $e \in Q$ is identity.

i.e $\forall a \in Q$, we have

$$a * e = e * a = a$$

Now

$a * e = a$	$e * a = a$
$\frac{ae}{3} = a$	$\frac{ea}{3} = a$
$e = 3 \in Q$	$e = 3 \in Q$

$\therefore$ Identity $e = 3 \in Q$ exist, hence $(Q, *)$ is monoid.

Ex-2

$\forall a, b \in N$; $a * b = a^b$,

then find identity ?

let $e \in N$ is identity

i.e $\forall a \in N$, we have

$$a * e = e * a = a$$

$a * e = a$	$e * a = a$
$\Rightarrow a^e = a$	$e^a = a$
$\therefore e = 1 \in N$	$e = a^{1/a} \notin N$ (∵ if $a \in N$ then $\frac{1}{a} \notin N$ ∴ $a^{1/a} \notin N$)
	$\therefore e \notin N$
Right identity exist (as e is on the right of binary operation)	Left identity exist (as e is on the left of binary operation)

* Identity exist only if,

Left identity exist and Right identity exist

and

Left identity = Right identity

Ex-3

$\forall a, b \in Q - \{-1\}$; $a * b = a + b + ab$

then find identity ?

Let 'e' is identity in Q,

$a * e = e * a = a$, $\forall a \in Q - \{-1\}$

$a * e = a$

$a + e + ae = a$

$e(1+a) = 0$

since $a \neq -1$, so $e = 0$

$e * a = a$

$e + a + ea = a$

$e(1+a) = 0$

here $a \neq -1$, $\therefore e = 0$

* ~~Identity exist~~

$\therefore$ Identity exist and $e = 0$.

Ex-4 let $M = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \Big/ a \neq 0 \text{ and } a \in \mathbb{R} \right\}$
then find identity w.r.t matrix multiplication

$$M = \left\{ \begin{bmatrix} b & b \\ b & b \end{bmatrix}, \begin{bmatrix} c & c \\ c & c \end{bmatrix}, \begin{bmatrix} e & e \\ e & e \end{bmatrix}, \begin{bmatrix} a & a \\ a & a \end{bmatrix} \right.$$

$$\left. a \neq 0, b \neq 0, c \neq 0, e \neq 0, \text{ and } a, b, c, e \in \mathbb{R} \right\}$$

Let $I = \begin{bmatrix} e & e \\ e & e \end{bmatrix}$ is the identity in M $\forall A \in M$,

$$A \cdot I = I \cdot A = A$$

$$A \cdot I = A$$

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} e & e \\ e & e \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$\begin{bmatrix} 2ae & 2ae \\ 2ae & 2ae \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$2ae = a$$

$$e = \frac{1}{2}$$

$$\therefore I = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

\# $I \cdot A = A$

$$\begin{bmatrix} e & e \\ e & e \end{bmatrix} \begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$\begin{bmatrix} 2ae & 2ae \\ 2ae & 2ae \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$2ae = a$$

$$e = \frac{1}{2}$$

$$I = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

21/4/21 Lecture 8B

Inverse Property

Let 'G' be a non empty set, with '*' is Binary operation such that $\forall a \in G, \exists b \in G$

$a * b = b * a = e$, where 'e' is identity, then 'b' is inverse of a.

It is denoted by $a^{-1} = b$.

Note:- If 'G' satisfies

i) Closure
ii) Associative
iii) Identity
iv) Inverse

with respect to '*'

then $(G, *)$ is called group.

Ex1: $\forall a, b \in Q$, $a * b = \frac{ab}{3}$, i) find inverse of 7?
ii) Is $(Q, *)$ group?

Let 'e' is identity,

$$a * e = e * a = a \quad \forall a \in G$$

now $a * e = a$

$$\frac{ae}{3} = a$$

$$\therefore e = 3$$

Let 'b' is inverse of 'a'.

$\therefore a * b = b * a = e \quad \forall a \in Q$

Let $a * b = e$

$\Rightarrow \frac{ab}{3} = 3 \quad (\because e = 3)$

$\Rightarrow b = \frac{9}{a}$

~~$\Rightarrow$~~ $\therefore a^{-1} = \frac{9}{a}$

$\therefore$ Inverse of '7' is $7^{-1} = \frac{9}{7}$

(No need to verify both eq^n $a * b = e$, $b * a = e$)

ii) Is $(Q, *)$ group?
We can not find inverse of '0' so $(Q, *)$ not a group.

Note:- In the above problem, if we consider $(Q_0, *)$ then $(Q_0, *)$ is group.
$(Q_0 = Q - \{0\})$

Commutative Property

$\forall a, b \in G$, if $a * b = b * a$

then 'G' is satisfying commutative w.r.t '$*$'.

* Abelian Group:-
A group which satisfies commutative property is called commutative group (or) Abelian group.

~~Gate~~ GATE-09

Q: Which one of the following is NOT necessarily property of a Group?

a) Commutativity
b) Associativity
c) Existence of inverse for every element
d) Existence of identity.

Sol a) Commutativity not necessary property of a Group.

GATE-13

Q A Binary operation $\oplus$ on a set of integers is defined as $x \oplus y = x^2 - y^2$. Which one of the following statement is TRUE about $\oplus$?

a) Commutative but not associative
b) Both commutative and associative
c) Associative but not commutative
d) Neither commutative nor associative.

$$x \oplus y = y \oplus x$$

$$x^2 - y^2 = y^2 - x^2 \quad \text{(NOT TRUE)}$$

And it is not associative also.

$\therefore$ option D → correct.

GATE-95

Q Let A be the set of all nonsingular matrices of order n×n over real numbers and let $*$ be the the matrix multiplication operator. Then

a) A is closed under $*$ but $\langle A, * \rangle$ is not a semigroup.
b) $\langle A, * \rangle$ is a semigroup but not a monoid.
c) $\langle A, * \rangle$ is a monoid but not a group.
d) $\langle A, * \rangle$ is a group but not an Abelian group.

M_1, M_2
~~A, B~~ ∈ Non singular matrices (M)

$M_1 * M_2 \in M$

∴ It satisfies closure property.

Associativity
$M_1(M_2 M_3) = (M_1 M_2) M_3$

Identity
$I_{n \times n}$

Inverse
Since matrix is non-singular therefore inverse exist.

Commutative
$M_1 M_2 \neq M_2 M_1$

∴ D - option correct

<A, *> is group but not Abelian group.

Algebraic Structure

A non empty set 'G' which equipped with one (or) more binary operations is called Algebraic Structure.
↓
satisfies closure property.

$\forall a, b \in N$, $a+b \in N$
$a \cdot b \in N$

$(N, +, \cdot)$ → Algebraic structure
$(N, +)$ → ''
$(N, \cdot)$ → ''

Every Quasigroup is called Algebraic structure but every Algebraic structure is not Quasigroup.

$(N, +, \cdot)$ → Not Quasigroup,

**☆

Order of a Group

i) No. of elements in a group 'G' is called order of a group 'G'. denoted by $O(G)$

ii) If $O(G) = n$ (finite), then 'G' is called finite group.

If $O(G) = \infty$, then 'G' is called infinite group.

Examples of finite groups

Group	Order	Identity	Inverse	Abelian
$(G=\{1\},\cdot)$	1	1	$1^{-1}=1$	Yes
$(G=\{1,-1\},\cdot)$	2	1	$1^{-1}=1$, ~~$(-1)\neq 1$~~ $(-1)^{-1}=-1$	Yes
$(G=\{1,\omega,\omega^2\},\cdot)$ where $\omega^3=1$	3	1	$1^{-1}=1$, $\omega^{-1}=\omega^2$, $(\omega^2)^{-1}=\omega$	Yes
$(G=\{1,-1,i,-i\},\cdot)$ where $i^4=1$	4	1	$1^{-1}=1$, $(-1)^{-1}=-1$ $(i)^{-1}=-i$, $(-i)^{-1}=i$	Yes
$(G=\{1,\omega,\omega^2,\omega^3,\omega^4\},\cdot)$ $\omega^5=1$	5	1	$1^{-1}=1$, $(\omega)^{-1}=\omega^4$ $(\omega^2)^{-1}=\omega^3$, $(\omega^4)^{-1}=\omega$ $(\omega^3)^{-1}=\omega^2$	Yes
Let S = Permutation group of {1, 2, 3} $S=\left\{\overset{S_1}{\begin{bmatrix}1&2&3\\1&2&3\end{bmatrix}}\overset{S_2}{\begin{bmatrix}1&2&3\\1&3&2\end{bmatrix}}\overset{S_3}{\begin{bmatrix}1&2&3\\2&3&1\end{bmatrix}}\overset{S_4}{\begin{bmatrix}1&2&3\\2&1&3\end{bmatrix}}\overset{S_5}{\begin{bmatrix}1&2&3\\3&1&2\end{bmatrix}}\overset{S_6}{\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}}\right.$ w.r.t function composition	6	S_1		No
$(G=\{1,\omega,\omega^2,\omega^3,\omega^4,\omega^5,\omega^6\},\cdot)$ where $\omega^7=1$	7	1	$1^{-1}=1$, $\omega^{-1}=\omega^6$, $(\omega^2)^{-1}=\omega^5$, $(\omega^3)^{-1}=\omega^4$ $(\omega^4)^{-1}=\omega^3$, $(\omega^5)^{-1}=\omega^2$	Yes
* Quaternian Group $(G=\{\pm 1,\pm i,\pm j,\pm k\},\cdot)$ $i^2=j^2=k^2=-1$ $i\cdot j=k$; $j\cdot i=-k$ $j\cdot k=i$, $k\cdot j=-i$ $k\cdot i=j$, $i\cdot k=-j$	8	1	$1^{-1}=1$, $(-1)^{-1}=-1$ $i^{-1}=-i$, $(-i)^{-1}=i$ $(j)^{-1}=-j$, $(-j)^{-1}=j$ $(k)^{-1}=-k$, $(-k)^{-1}=k$	No

1:08:50

$(G=\{1,-1\},\cdot)$

Cayley Table

	1	-1
1	1*	-1
-1	-1	1*

Identity = 1

$1^{-1}=1$

$(-1)^{-1}=-1$

* Closure :- $\forall\, a,b\in G$, $a\cdot b\in G$
(if all elements of composition ~~exist~~ table exist in given set then it is called closure).

Number system always ~~verify~~ ~~follow~~ satisfy. associative Property

Inverse:
$\forall a \in G, \ \exists b \in G$
such that $a * b = b * a = e$

Identity:
$\forall a \in G, \exists e \in G$
$a * e = e * a = a$

Prove that $(G = \{1, \omega, \omega^2\}, \cdot)$ is abelian group $(\omega^3 = 1$

$\cdot$	1	ω	ω^2
1	1^*	ω	ω^2
ω	ω	ω^2	1^*
ω^2	ω^2	1^*	ω

*1st Row = 1st column
*2nd Row = 2nd column

$\omega^4 = \omega^3 \cdot \omega$
$= 1 \cdot \omega = \omega$

Identity element $= 1$

$1^{-1} = 1$
$\omega^{-1} = \omega^2$
$(\omega^2)^{-1} = \omega$

$*$ $\{1, \omega, \omega^2, ----, \omega^r, ---\omega^{p-1}\}$

where $\omega^p = 1$, here 'p' is prime no.
is always abelian group
Identity element $= 1$
Inverse $(\omega^r)^{-1} = \omega^{p-r}$

$S = \{1, 2, 3\}$

#

$$G = \left\{ \overset{S_1}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}} \overset{S_2}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}} \overset{S_3}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}} \overset{S_4}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}} \overset{S_5}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}} \overset{S_6}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}} \right\}$$

$f \circ g(x) = f(g(x))$

$S_1 \circ S_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

	S_1	S_2	S_3	S_4	S_5	S_6
S_1	S_1	S_2	S_3	S_4	S_5	S_6
S_2						
S_3						
S_4						
S_5						
S_6						

$S_1 \circ S_1(1) = S_1(S_1(1))$
$= S_1(1) = 1$

$S_1 \circ S_1(2) = S_1(S_1(2))$
$= S_1(2) = 2$

$S_1 \circ S_1(3) = S_1(S_1(3)) = S_1(3) = 3$

① Upto group of order 5 every group is abelian

② The least non abelian group is of order 6, which is permutation group.

③ Every group of prime order is always abelian.

④ In a group identity element is always unique

⑤ In a group every element has unique inverse.

GATE-04

Q The following is the incomplete operation table of a 4 element group.

*	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c		
c	c	e	~~e~~ a	b

The last row of the table is

a) c a e b b) c b a e
c) c b e a d) c e a b

4 element group → abelian group
↓ symmetric

i.e 1st row = 1st coloum
2nd " = 2nd "

a) b) c) → wrong option
d → correct option

Now,
$c*b = a$ (or) $\underbrace{c*b = b}_{\text{Not Possible}}$ (c & e already in the row)

(If it is possible then 'e' is identity but we have 'e' is identity and it always unique.)

$\therefore$ $c*b = a$

GATE-07

Q How many different non-isomorphic Abelian groups of order 4 are there?
different properties

a) 2 b) 3 c) 4 d) 5

Isomorphic → same properties

2:02:05

- let $G_1 = \{e, a, b, c\}$

  $e^{-1} = e, \; a^{-1} = a, \; b^{-1} = b, \; c^{-1} = c$

- let $G_2 = \{e, a, b, c\}$

  $e^{-1} = e, \; a^{-1} = b, \; b^{-1} = a, \; c^{-1} = c$

- $G_3 = \{e, a, b, c\}$

  $e^{-1} = e, \; a^{-1} = c, \; b^{-1} = b, \; c^{-1} = a$

- $G_4 = \{e, a, b, c\}$

  $e^{-1} = e, \; a^{-1} = a, \; b^{-1} = c, \; c^{-1} = b$

$G_1 \rightarrow$ every element has self inverse

$G_2, G_3, G_4 \rightarrow$ Two element has self inverse
other two elements has
inverse with each other

$G_2 \cong G_3 \cong G_4$

↓

Isomorphic sign

No. of Non-isomorphic abelian group = 2

A → correct Answer

Examples for Infinite groups

1) $(\mathbb{Z}, +), (\mathbb{R}, +), (\mathbb{Q}, +), (\mathbb{C}, +)$
are infinite abelian group.

- Prove that $(\mathbb{Z}, +)$ is an infinite abelian group?

1) Closure :- $\forall\, a, b \in \mathbb{Z}, \quad a + b \in \mathbb{Z}$

2) Associativity :

$\forall\, a, b, c \in \mathbb{Z}, \quad a + (b + c) = (a + b) + c$

3) Identity :-

$\forall\, a \in \mathbb{Z}, \; \exists\, 0 \in \mathbb{Z}$ such that

$$a + 0 = 0 + a = a$$

identity element = 0

4) Inverse :

$\forall\, a \in \mathbb{Z}, \; \exists\, -a \in \mathbb{Z}$, such that

$$a + (-a) = (-a) + a = 0$$

$$\therefore a^{-1} = -a$$

5) Commutative :

$\forall\, a, b \in \mathbb{Z}, \quad a + b = b + a$

$\therefore (\mathbb{Z}, +)$ infinite abelian group

Q Is $(Q, \cdot)$ abelian group?

Closure :-
$\forall a, b \in Q$, $a \cdot b \in Q$

Associative :-
$\forall a, b, c \in Q$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Identity :-
$\forall a \in Q$, $\exists 1 \in Q$,

$$a \cdot 1 = 1 \cdot a = a$$

$\therefore$ Identity is 1.

iv) Inverse :-
$\forall a \in Q$, $\exists \frac{1}{a} \in Q$, ~~a~~

~~$a \cdot \frac{1}{a} \neq 1$~~

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$$

$$\therefore a^{-1} = \frac{1}{a}$$

Clearly, $0^{-1} = \frac{1}{0}$ undefined

$\therefore$ Inverse of '0' does not exist

$\therefore$ $(Q, \cdot)$ is not a group.

Note:-

~~1) $Q_0, \cdot$~~

1) $(Q_0, \cdot)$ is infinite abelian group.

2) $(R_0, \cdot)$ is infinite abelian group.

3) $(C_0, \cdot)$ is infinite abelian group.

21/4/21

Lecture-9A

Addition & Multiplication Modulo

$3 +_4 8 = \boxed{3} \rightarrow$ remainder

$5 +_3 9 = 2$

$a +_m b = r \ (0 \leq r < m)$

Addition modulo of a & b under 'm'.

Addition Modulo 'm' of a, b :-

i) It is denoted by $a +_m b$

ii) $a +_m b = r \quad (0 \le r < m)$,

'r' is non-negative ~~number~~ remainder which is obtained when (a+b) is divided with 'm'.

~~$3 +_m 8 =$~~ $3 +_4 8 = 3$

$5 +_3 9 = 2$

$a +_m b = r \quad (0 \le r < m)$

Multiplication modulo 'm' of a, b :-

$a \times_m b = r \; (0 \le r \le m)$

$8 \times_3 5 = 1$

$7 \times_2 8 = 0$

Q Which of the following are valid groups ?

1.) $\{0, 1, 2\}_{+3}$ 2.) $\{0, 1, 2\}_{\times 3}$

$+_3$	0	1	2
0	0*	1	2
1	1	2	0*
2	2	0*	1

Composition Table containing only remainders

identity = 0

$0^{-1} = 0$ $0^{-1} = 3-0/3 = 0$

$1^{-1} = 2$ $1^{-1} = 3-1/3 = 2$

$2^{-1} = 1$ $2^{-1} = 3-2/2 = 1$

∴ This is abelian group

Note:-
$Z_m = \{0, 1, 2, 3, \ldots, m-1\}_{+m}$
is always abelian group

identity = 0 ; $r \in Z_m$, ~~$r^{-1} = (m-r)$~~

$r^{-1} = (m-r) \bmod m$

Divide (m-r) by m and write the remainder

ii) $\{0, 1, 2\}_{\times 3}$

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

← 1 is not there

$\rightarrow \frac{2\times 2}{3} = \frac{4}{3} = \frac{1}{1}$ Remainder

Identity element = 1

0^{-1} = does not exist

∴ it is not a group

iii) $A = \{1, 2, 3\}_{\times 4}$

	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

$\rightarrow \notin A$

∴ Closure not satisfied.

Note:-

$Z_p = \{1, 2, 3, ---, p-1\}_{\times p}$

where p = prime number

∴ Z_p is an abelian

Ex:- $Z_5 = \{1, 2, 3, 4\}_{\times 5}$ is an Abelian Group

GATE

Q The set $\{1, 2, 4, 7, 8, 11, 13, 14\}$ is a group under multiplication modulo 15. The inverse of 4 and 7 are respectively.

a) 3 and 13
b) 2 and 11
c) 4 and 13
d) 8 and 14

Identity element = 1

$4 \times_{15} \_\_ = 1$

Verifying from the option

$4 \times_{15} 4 = 1$ ✓ , $7 \times_{15} 13 = 1$

∴ C → option

GATE-06

Q The set $\{1, 2, 3, 5, 7, 8, 9\}$ under multiplication modulo 10 is not a group. Given below are four possible reasons. Which one of them is FALSE?

a) It is not closed
b) 2 does not have an inverse
c) 3 does not have an inverse
d) 8 does not have an inverse

Identity = 1

a → TRUE $(3 \times 2 = 6 \rightarrow$ not in set$)$

b → TRUE

c → FALSE

$3 \times \underset{10}{7} = 1$

$\therefore\ 3^{-1} = 7$ (inverse exist)

d → TRUE

∴ c → correct option

Order of an element in a Group

Let $(G, \cdot)$ be a group, for $a \in G$, the order of 'a' is denoted by $O(a)$ and it can be defined ~~by~~ as $O(a) = n$, where 'n' is least +ve integer such that $a^n = e$ (here 'e' is identity)

Note:- In $(G, +)$, if $\exists$ a least +ve integer 'n' such that $na = e$ then $O(a) = n$.

Q $(G = \{1, \omega, \omega^2\}, \cdot)$; $\omega^3 = 1$

here $e = 1$ $\left(\because a^n = e \Rightarrow O(a) = n\right)$

Order of 1

$(1) = 1 \Rightarrow O(1) = 1$

Order of 2

$(\omega)^3 = 1 \Rightarrow O(\omega) = 3$

Order of ω^2

$(\omega^2)^3 = 1 \Rightarrow O(\omega^2) = 3$

Q $(G = \{1, -1, i, -i\}, \cdot)$ $i^2 = -1$

here $e = 1$

Order of 1

$(1)^1 = 1$ $\therefore O(1) = 1$

Order of -1

$(-1)^2 = 1$ $\therefore O(-1) = 2$

Order of i

$(i)^4 = 1$ ∴ $O(i) = 4$

Order of $-i$

$(-i)^4 = 1$ ∴ $O(-i) = 4$

Note:- (Important Points)

i) $O(e) = 1$ [element other than e must have order > 1

ii) $O(a) = O(a^{-1})$

iii) $O(a)$ must divides $O(G)$

iv) let $O(a) = n$, and ∃ a positive integer 'm' such that

$a^m = e$, then 'n' divides 'm'.

22/4/21 1:09:44

Pg 28

Q.73 Assume 'g' is an element of group 'G', Consider the following conditions of 'g' with 'e' as identity element.

i) $g^8 = e$ ii) $g^2 \neq e$ iii) $O(g) \neq 8$

what is the order of 'g'.

Sol ∃ a least +ve integer 'n' such that

$g^n = e \Rightarrow O(g) = n$

but it is given that $g^8 = e$

∴ n divides 8.

∴ $n = 1$ or 2 or 4 or 8

But only identity element can have order 1 and 'g' is not identity element i.e. $n \neq 1$

ATQ

~~$n \neq 4$ and~~

$n \neq 2$ and $n \neq 8$.

∴ $n = 4$

Subgroup:

let $(G, \cdot)$ be a group.

A non-empty subset of 'H' of 'G' is called Subgroup of 'G' if $(H, \cdot)$ itself forms a group.

Subgroup must contain identity element.

1) $(G = \{1, -1, i, -i\}, \cdot)$

$O(G) = 4$, Order of subgroup : 1 (or) 2 (or) 4

$H_1 = \{1\} \subseteq G$ and $(H_1, \cdot)$ forms group

$\Rightarrow H_1$ is subgroup of 'G'.

$H_2 = \{1, -1\} \subseteq G$ and $(H_2, \cdot)$ forms group

$\therefore H_2$ is a subgroup of 'G'.

$H_3 = \{1, i\} \subseteq G$ and $(H_3, \cdot)$ is ~~no~~ not group

H_3 is not subgroup.

$\cdot$	1	i
1	1	i
i	i	$-1 \notin H_3$ (∵ closure property not satisfied)

$H_4 = \{1, -1, i, -i\} \subseteq G$, and

$(H_4, \cdot)$ is a subgroup.

Lagrange's Theorem

If 'H' is a subgroup of a finite group 'G' then $O(H)$ divides $O(G)$.

Note: Converse of Lagrange's Theorem need not be true.

i.e if $O(H)$ divides $O(G)$ then H need not be subgroup of 'G'.

$(G = \{1, -1, i, -i\}, \cdot)$ $\therefore O(G) = 4$

$(H = \{1, i\}, \cdot)$, $O(H) = 2$

Clearly $O(H)$ divides $O(G)$

but H is not a subgroup of 'G'

Q Find all subgroups of the following groups.

Ex1 $(G = \{1, -1\}, \cdot)$ $O(G) = 2$

$O(\text{subgroup}) = 1 \text{ (or) } 2$

$(H_1 = \{1\}, \cdot)$

$(H_2 = \{1, -1\}, \cdot)$

Ex-2 $(G = \{1, \omega, \omega^2\}, \cdot) \rightarrow O(G) = 3$

$O(\text{subgroup}) = 1 \text{ or } 3$

$(H_1 = \{1\}, \cdot)$

$(H_2 = \{1, \omega, \omega^2\}, \cdot)$

Ex3 $(G = \{\pm 1, \pm i, \pm j, \pm k\}, \cdot)$

$O(G) = 8$

$O(\text{subgroup}) = 1 \text{ or } 2 \text{ or } 4 \text{ or } 8$

$H_1 = \{1\}$

$H_2 = \{\pm 1, \pm j \pm j, \pm k\}$

Improper subgroups (Trivial subgroups)

$H_3 = \{1, -1\}$

$H_3 = \{1, -1\}$

$H_4 = \{1, -1, i, -i\}$

$H_5 = \{1, -1, j, -j\}$

$H_6 = \{1, -1, K, -K\}$

Proper Subgroups (Non-Trivial Subgroups)

GATE-18

Q Let G be a finite group on 84 elements. The size of a largest possible proper subgroup of G is ____

Sol $O(\text{Subgroup})$ divides $O(G)$

~~42 is the largest~~

$O(\text{Subgroup}) = 42$ is the largest possible proper subgroup.

* $O(G) = 84$

proper subgroup

$O(H) = 1, 2, -----, 42, 84$

Improper subgroup

42 → Answer

GATE - 14 - set 3

Q Let G be a group with 15 elements.
Let L be a subgroup of G.
It is known that $L \neq G$ and that
the size of L is atleast 4. The
size of L is ______.

$O(G) = 15$

$O(L) = 1$ (or) 3 (or) 5 (or) 15

$L \neq G$ (15 not possible)

and L is atleast 4

$\therefore$ $L = 5$ (Answer)

Note:- $n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times p_3^{\alpha_3} \times \cdots \times p_n^{\alpha_n}$

where $p_1, p_2, p_3, \ldots p_n$ are distinct prime numbers.

& $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \mathbb{N}$

i) $\phi(n)$ = No. of +ve integers which are less than 'n' and coprime to 'n'.

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_n}\right)$$

ii) $\tau(n)$ = No. of +ve divisors of 'n'

$$= (1+\alpha_1)(1+\alpha_2) \cdots (1+\alpha_n)$$

Let $n = 12 = 2^2 \times 3^1 = p_1^{\alpha_1} \times p_2^{\alpha_2}$

here $p_1 = 2$, $p_2 = 3$

$\alpha_1 = 2$, $\alpha_2 = 1$

$\phi(12)$ = No. of +ve integers which are less than 12 and coprime to 12

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_n}\right)$$

$$= 12\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)$$

$$= 12 \times \frac{1}{2} \times \frac{2}{3}$$

$$= 4$$

H.C.F or G.C.D

$(1, 12) = 1$
$(2, 12) \neq 1$
$(3, 12) \neq 1$
$(4, 12) \neq 1$
$(5, 12) = 1$
$(6, 12) \neq 1$
$(7, 12) = 1$
$(8, 12) \neq 1$
$(9, 12) \neq 1$
$(10, 12) \neq 1$
$(11, 12) = 1$

$\{1, 5, 7, 11\}$

4

~~$\{1, 5, 7, 11\}$~~

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$\tau(12) =$ No. of +ve divisors of 12

$= (1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_n)$

$= (1+2)(1+1)$

$= 6$

$$\left[\begin{array}{c} 1, 2, 3, 4, 6, 12 \\ \therefore \tau(12) = 6 \end{array}\right]$$

$G = \{\pm 1, \pm i,$

i) $(G = \{1, -1\}, \cdot)$: $\rightarrow O(H_1) = 1$

$(H_1 = \{1\}, \cdot)$, $(H_2 = \{1, -1\}, \cdot)$

ii) $(G = \{1, \omega, \omega^2\}, \cdot)$: $\rightarrow O(H_2) = 2$

$(H_1 = \{1\}, \cdot)$; $(H_2 = \{1, \omega, \omega^2\}, \cdot)$

iii) $(G = \{1, -1, i, -i\}, \cdot)$: $O(G_3) = 4$

$O(H_1) = 1$ $\quad O(H_2) = 2$

$(H_1 = \{1\}, \cdot)$, $(H_2 = \{1, -1\}, \cdot)$

$O(H_3) = 4$

$(H_3 = \{1, -1, i, -i\}, \cdot)$

iv) $G = \{\pm 1, \pm i, \pm j, \pm k\}$:

$O(H_1) = 1$ $\quad O(H_2) = 2$

$(H_1 = \{1\}, \cdot)$, $(H_2 = \{1, -1\}, \cdot)$

$(H_3 = \{1, -1, i, -i\}, \cdot)$, $(H_4 = \{1, -1, j, -j\}, \cdot)$

$(H_5 = \{1, -1, k, -k\}, \cdot)$ $O(H_3) = O(H_4) = O(H_5) = 4$

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$(H_6 = \{\pm 1, \pm i, \pm j, \pm k\}, \cdot)$ $O(H_6) = 8$

No. of Distinct ordered subgroups $= \{1, 2, 4, 8\} = 4$

Note:- If $O(G) = n$, then number of distinct ordered subgroups of $G = \tau(n)$

11:20

Cyclic Group

(With respect to multiplication)

Let $(G, \cdot)$ be a group

$G = \{a^n / n \in Z\}$

then 'G' is called cyclic group generated by 'a'. Here 'a' is called generator.

$\therefore G = \langle a \rangle$

With respect to addition

Let $(G, +)$ be a group

$G = \{na / n \in Z\}$

then 'G' is called cyclic group generated by 'a'. Here 'a' is called generator

$\therefore G = \langle a \rangle$

$(G = \{1, -1.\}, \cdot)$

Ex1 $(G = \{1, -1\}, \cdot)$

$(1)^0 = 1$, $(1)^1 = -1$ ⟹ $G \neq \langle 1 \rangle$

$(-1)^0 = 1$, $(-1)^1 = -1$ ⟹ $G = \langle -1 \rangle$

∴ 'G' is a cyclic group with generator −1.

Ex2 which of the following are cyclic?

1) $(G = \{1, \omega, \omega^2\}, \cdot)$

$(1)^0 = 1$, $(1)^1 \neq \omega$, $(1)^1 \neq \omega^2$

∴ $G \neq \langle 1 \rangle$

$(\omega)^0 = 1$; $(\omega)^1 = \omega$; $(\omega)^2 = \omega^2$

∴ $G = \langle \omega \rangle$

$(\omega^2)^0 = 1$, $(\omega^2)^2 = \omega$, $(\omega^2)^1 = \omega^2$

∴ $G = \langle \omega^2 \rangle$

∴ 'G' is called cyclic

Properties

ii) ~~$(G = \{1, -1, i, -i\}, \cdot)$~~

Note:- i) If $O(G) = 1$, then $G = \langle e \rangle$

ii) If $O(G) > 1$, then $G \neq \langle e \rangle$

iii) If $G = \langle a \rangle$ then $G = \langle a^{-1} \rangle$

iv) If 'G' is cyclic and $O(G) = n$ then no. of generators $= \phi(n)$

ii) $(G = \{1, -1, i, -i\}, \cdot)$

$O(G) = 4\ (>1)$ ∴ $G \neq \langle 1 \rangle$.

$(-1)^1 \neq i$ ∴ $G \neq \langle -1 \rangle$

$(i)^0 = 1$, $(i)^1 = i$, $(i)^2 = -1$, $(i)^3 = -i$

∴ $G = \langle i \rangle$

∴ $G = \langle -i \rangle$ $(\because i^{-1} = i)$

∴ ~~G≠~~ G is cyclic

(iv)th property

∴ $O(G) = 4 = 2^2 = p_1^{\alpha_1}$

$\phi(4) = 4\left(1 - \frac{1}{2}\right) = 2$

∴ 2 generator

These properties are respect to multiplication

Properties of a Group

if addition operation came then replace (· with +)

Let 'G' be a group with binary operation *

i) The identity element in a group is unique.

ii) The inverse of each element of a group is unique.

iii) The inverse of 'a' is a^{-1} and $(a^{-1})^{-1} = a$.

iv) $\forall a, b \in G$, then $(ab)^{-1} = b^{-1}a^{-1}$

v) $\forall a, b, c \in G$, if $a \cdot b = a \cdot c$ then $b = c$ by using cancellation law.

vi) If a, b are any two elements of a group G then the equation $ax = b$ and $ya = b$ have unique solution in G, given by $x = a^{-1}b$ and $y = ba^{-1}$ ($y = ba^{-1}$) respectively.

$$ax = b$$
$$a^{-1} \cdot a x = a^{-1} \cdot b$$
$$e \cdot x = a^{-1} b$$
$$x = a^{-1} b$$

Properties of a Cyclic Group

i) Every cyclic group is an abelian group

ii) If a is the generator of cyclic group 'G' then a^{-1} is also generator of cyclic group G.

iii) If a finite group of order 'n' contains elements of order 'n' then that group must be cyclic. [$G = \{1, \omega, \omega^2\}$ $o(G) = 3$ $o(\omega) = 3$]

iv) Every group of prime order is cyclic.

v) Every subgroup of cyclic group is also cyclic.

Properties of a Subgroup

i) A necessary and sufficient condition for a non-empty subset H of a group to be subgroup is that
$a \in H, b \in H \Rightarrow a \cdot b^{-1} \in H$

ii) A necessary and sufficient condition for a non-empty finite subset H of a group G to be a subgroup is

i) $a \in H, b \in H \Rightarrow ab \in H$
ii) $a \in H \Rightarrow a^{-1} \in H$

iii) If H, K are two subgroups of a group 'G' then HK is a subgroup of G iff $HK = KH$.

iv) If H, K are two subgroups of a group 'G' then $H \cap K$ is also subgroup of G.

v) If H, K are two subgroups of group 'G' then $H \cup K$ need not be subgroup of G.

$$G = \{\pm 1, \pm i, \pm j, \pm k\}$$

$$H = \{1, -1, i, -i\}$$

$$K = \{1, -1, j, -j\}$$

are subgroups of G.

but $H \cup K = \{1, -1, i, -i, j, -j\}$ is not subgroup.

$O(H \cup K) = 6$; $O(G) = 8$

6 not divide 8

36;43

GATE-2021

Q. Let 'G' be a group of order '6' and H is subgroup of 'G' such that $1 < |H| < 6$. which of the following options are correct?

Ⓐ Both 'G' and 'H' are always cyclic
Ⓑ G is always cyclic but H may not be cyclic
Ⓒ G may not be cyclic but 'H' is always cyclic
Ⓓ Both 'G' and 'H' may not be cyclic.

$$O(G) = 6$$

$$O(H) = \cancel{1 / (or)}\ 2\ (or)\ 3\ \cancel{(or\ 6)}$$

since $O(H) = 2$ (or) 3 which is prime order.

Hence 'H' is cyclic.

∴ G may not be cyclic but 'H' is always cyclic.

∴ C → Correct option

GATE - 09

Q For the composition table of a cyclic group

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

Which one of the following choices is correct?

a) a, b are generators
b) b, c are generators
c) c, d are generators
d) a, d are generators

$O(G) = 4 = 2^2 = p_1^{\alpha_1}$

No. of generators $= \phi(4)$
$= 4\left(1 - \frac{1}{2}\right)$
$= 2$

$O(G) = 4 \ (>1)$

$\therefore$ identity element cannot be generator

Identity element = a

i.e. $G \neq \langle identity \rangle \neq \langle a \rangle$

here $b^{-1} = b$, $c^{-1} = d$, $d^{-1} = c$

If we consider b as generator then we will get only 1 generator, but we expect 2 generator

$\therefore$ c is generator and its inverse d is also an generator

$\therefore$ C → option correct.

GATE 2021

Q Consider the following sets, where $n \geq 2$

S_1: Set of all $n \times n$ matrices with entries from the set $\{a, b, c\}$

S_2: Set of all functions from the set $\{0, 1, 2, \ldots, n^2 - 1\}$ to the set $\{0, 1, 2\}$

$$\begin{bmatrix} & \\ & \end{bmatrix}_{n \times n}$$

Total no. of elements (position) = n^2

Each ~~element~~ position has 3 possibilites

$\therefore$ Total no. of Matrices $= 3^{n^2}$

S_2: No. of functions possible $= 3^{n^2} = |B|^{|A|}$

∴ $|S_1| = |S_2|$

~~∴ It is bijective~~

∴ There exist a bijection from S_1 to S_2.

~~∴ option D~~

Options of p. last problem

1.) There exist a surjection from S_1 to S_2
2.) There does not exist an injection from S_1 to S_2
3.) There does not exist a bijection from S_1 to S_2
4.) There exist a bijection from S_1 to S_2.

Since Bijection exist
∴ surjection also exist from S_1 to S_2

∴ A, D options correct

Combinatorics 1987 - 2020

1) Principle of mutual inclusion and exclusion - 5Q
2) Euler's function - 4Q
3) Sum Rule, Product Rule - 5Q
4) Permutation and combination - 8Q
5) Pigeon hole Principle - 3Q
6) Derangement - 2Q
7) Recurrence Relations - 20Q
8) Generating Functions - 5Q

Recursion (8 marks)

i) Programming

ii) Data structure
- Stack
  - Recursion

iii) Algorithm
- Divide - Conquer master theorem for solving Recursion Eqn

iv) Dmaths
- Combinatorics
  - Recurrence Relations

v) Compiler Design
- Implementation of Recursion

23|4|2

Euler's Function

1.) It is denoted by $\phi(n)$:

$\phi(n)$ = No. of +ve integer which are less than 'n' and coprime to 'n'.

$$= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)------\left(1-\frac{1}{p_n}\right)$$

$\because$ Let $n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times --- \times p_n^{\alpha_n}$

where $p_1, p_2, --, p_n$ are distinct primes

$\alpha_1, \alpha_2, -----, \alpha_n \in N$

✱ Properties of Euler's function

$$n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times --- \times p_n^{\alpha_n}$$

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)---\left(1-\frac{1}{p_n}\right)$$

1.) $\phi(p) = p-1$ ($\because$ here 'p' is prime)

Ex:- $\phi(31)=30$ [31 is prime no]

2.) $\phi(m \cdot n) = \phi(m) \times \phi(n)$, where $G \cdot C \cdot D(m,n) = 1$, $H \cdot C \cdot F(m,n) = 1$

$\phi(6) = \phi(2 \times 3)$:

$= \phi(2) \times \phi(3)$ $G \cdot C \cdot D(2,3) = 1$

$= \cancel{\phi}(2-1) \times (3-1)$

$= 2$

3) $\phi(p^n) = p^n - p^{n-1}$

where 'p' is prime no.

$\therefore \phi(8) = \phi(2^3)$

$= 2^3 - 2^{3-1}$

$= 8 - 4$

$= 4$

★ Euler's Theorem

1) $a^{\phi(m)} \cong 1 \pmod{m}$ [When we divide $a^{\phi(m)}$ by m then the remainder is 1]

where $G \cdot C \cdot D(a,m) = 1$

Ex:- $2^{\phi(3)} \equiv 1 \pmod{3}$,

$2^{3-1} \equiv 1 \pmod{3}$ $(\because \phi(3) = 3-1)$

2) ~~$a^{p-1} \cong 1 \text{ mod}$~~

$a^{p-1} \cong 1 \pmod{p}$, where 'p' is prime no. and $(G \cdot C \cdot D)(a,p) = 1$, $H \cdot C \cdot F(a,p) = 1$

22:22

Q Numbers of positive integers which are less than 1368 and coprime to 1368 is ______.

$1368 = 2^2 \times 3^3 \times 19^1$

$\phi(1368) = 1368\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{19}\right)$

$= 432$

Q Number of positive integers which are less than 317 and coprime to 317 is ______.

317 → prime no.

$\phi(p) = p - 1$

$= 317 - 1$

$\phi(p) = 316$

GATE

Q The formula for the number of positive integers m which are less than p^K and relatively prime to p^K, where p is a prime number and K is a positive integer is ______.

coprime ←

a) $p^K(p-1)$ b) $p^{K-2}(p-1)$
c) $p^K(p-2)$ d) $p^{K-1}(p-1)$

$$\phi(p^K) = p^K - p^{K-1}$$

$$= p^{K-1}(p-1)$$

∴ D → correct option

GATE - 15 - set 2

Q The no. of divisors of ~~2000~~ 2100 is ______.

$$2100 = 7 \times 3 \times (100)$$

$$= 7 \times 3 \times 2^2 \times 5^2$$

No. of Divisors of 2100 $= (1+2)(1+2)(1+1)(1+1)$

$$= 3 \times 3 \times 2 \times 2$$

$$\tau(n) = 36$$

∴ No. of divisor of 2100 = 36

$$n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times \dots \times p_n^{\alpha_n}$$

$$\tau(n) = (1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_n)$$

GATE-05

Q Let $n = p^2 q$, where p and q are distinct prime numbers. How many numbers m satisfy $1 \le m \le n$ and G.C.D$(m, n) = 1$?

a) $p(q-1)$
b) pq
c) $(p^2-1)(q-1)$
d) $p(p-1)(q-1)$

$$n = p^2 q$$

$$\phi(n) = p^2 q\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)$$

$$= p(p-1)(q-1)$$

∴ D → correct option

GATE-08

Q The exponent of 11 in the prime factorization of 300! is

a) 27 b) 28
c) 29 d) 30

$$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$(2 \times 5) \times (3 \times 3) \times (2 \times 2 \times 2) \times 7 \times (3 \times 2) \times 5 \times (2 \times 2) \times 3 \times 2 \times 1$$

$$= 2^8 \times 3^4 \times 5^2 \times 7^1$$

In (10!)

i) Exponent of 2 = 8
ii) " " of 3 = 4
iii) " " 5 = 2
iv) " " 7 = 1

2 | 10
2 | 5
2 | 2 → 5+2+1 = 8
 1

3 | 10
3 | 3 → 3+1 = 4
 1

5 | 10
 2 → 2

Similarly

Exponent of 11 in 300!

11 | 300
11 | 27
 | 2

27 + 2 = 29

∴ C → correct option

GATE-14-set2

divisors

Q The no. of distinct positive integeral factor of 2014 is ____.

$\tau(2014) = 2 \times 1007$

GATE-17-set1

Q The value of the expression 13^{99} (mod 17), in the range 0 to 16 is ____.

$a^{\phi(m)} \cong 1 \pmod m$ (Euler's Theorem)

OR

$a^{\phi(p)} \cong 1 \pmod p$

$a^{p-1} \cong 1 \pmod p$

where G.C.D $(a,p) = 1$

$13^{16} \cong 1 \pmod{17}$ → as it is 1, so any power of 13^{6} also leave remainder 1 when divided by 17

$13^{99} \bmod 17 \cong (13^{96} \times 13^{3}) \bmod 17$

$\cong (13^{96} \bmod 17) \times (13^{3} \bmod 17)$

Condition → G.C.D $(13^{3}, 17) = 1$

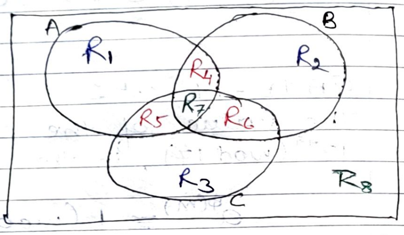
$\cong ((13^{16})^{6} \bmod 17) \times (13^{3} \bmod 17)$

$= 1 \times 2197 \bmod 17$

$= 4$ (Remainder)

→ By euler's Theorem G.C.D $(13, 17) = 1$
Remainder = 1

Principle of Inclusion and Exclusion



Let A, B and C are three different types of skills.

1) No. of Persons who are having only skill 'A' $= n(A \cap \bar{B} \cap \bar{C}) = R_1$

2) Only skill 'B' $= n(\bar{A} \cap B \cap \bar{C}) = R_2$

3) Only skill 'C' $= n(\bar{A} \cap \bar{B} \cap C) = R_3$

4) Exactly one skill $= R_1 \cup R_2 \cup R_3$

5) No. of Persons who are having only skill A and skill B $= n(A \cap B \cap \bar{C}) = R_4$

6) Only skill A and skill C $= n(A \cap \bar{B} \cap C) = R_5$

7) Only skill B and skill C $= n(\bar{A} \cap B \cap C) = R_6$

8) Exactly two skills $= R_4 \cup R_5 \cup R_6$.

9) Atmost two skills
$= R_8 \cup R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup R_6$

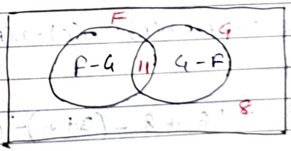
10) Exactly Three skills $= n(A \cap B \cap C) = R_7$

11) Atleast Two skills $= R_4 \cup R_5 \cup R_6 \cup R_7$

12) None of the skills $= R_8 = n(\overline{A \cup B \cup C})$

Q In a class of 50 students, 11 students can speak both french and German and 8 students can speak neither German nor french. No. of students who can speak only one of the two languages is ______.

S

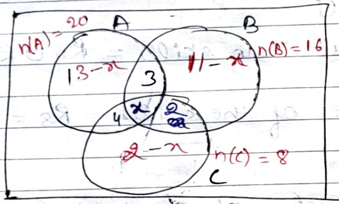


$n(F-G) + n(G-F) + 11 + 8 = 50$

$n(F-G) + n(G-F) = 31$

$\therefore$ 31 → Ans

Q A, B and C are 3 sets such that
$n(A \cup B \cup C) = 31$, $n(A) = 20$, $n(B) = 16$,
$n(C) = 8$, $n(A \cap B \cap \bar{C}) = 3$,
$n(A \cap \bar{B} \cap C) = 4$, $n(\bar{A} + B + C) = 2$.
then $n(A \cap B \cap C) =$ ______ ?



$(13-x) + (11-x) + (2-x) + 3 + 4 + 2 + x = 31$

$35 - 2x = 31$

$2x = 4$, $\therefore x = 2$

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$
$- n(A \cap C) + n(A \cap B \cap C)$

$= 20 + 16 + 8 - (3+x) - (4+x) - (2+x) + x$

$31 = 35 - 2x$

$x = 2$

Q No. of integers between 1 to 10 inclusive which are divisible by 2, 3, or 5

$1 —— 10 \begin{Bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{Bmatrix}$

$n(2) = \left[\frac{10}{2}\right] = 5$ $([\cdot] = GIF)$

$n(3) = \left[\frac{10}{3}\right] = 3$

$n(5) = \left[\frac{10}{5}\right] = 2$

$n(2 \cap 3) = \left[\frac{10}{L.C.M(2,3)}\right] = 1$

$n(2 \cap 5) = \left[\frac{10}{L.C.M(2,5)}\right] = \left[\frac{10}{10}\right] = 1$

$n(3 \cap 5) = \left[\frac{10}{L.C.M(3,5)}\right] = 0$

$n(2 \cap 3 \cap 5) = \left[\frac{10}{L.C.M(2,3,5)}\right]$

$= 0$

$n(2 \cup 3 \cup 5) = n(2) + n(3) + n(5)$

$- n(2 \cap 3) - n(2 \cap 5) - n(3 \cap 5)$

$+ n(2 \cap 3 \cap 5)$

$= 10 - 2 + 0$

$n(2 \cup 3 \cup 5) = 8$

Q The numbers of integers between 1 and 400 inclusive that are not divisible by 5, 6 or 8 is

$n(5) = \left[\frac{400}{5}\right] = 80$

$n(6) = \left[\frac{400}{6}\right] = 66$

$n(8) = \left[\frac{400}{8}\right] = 50$

$n(5 \cap 6] = \left[\frac{400}{30}\right] = 13$

$n(5 \cap 8] = \left[\frac{400}{40}\right] = 10$

$n(6 \cap 8) = \left[\frac{400}{24}\right] = 16$

196

$\frac{24}{5}$ 120

$n(5 \cap 6 \cap 8) = \left[\frac{400}{120}\right] = 3$

4 45 270

$n(5 \cup 6 \cup 8) = 80 + 66 + 50 - 13 - 10 - 16 + 3$

$= 160$

∴ Not divisible by 5, 6, 8

$\therefore n(\overline{5 \cup 6 \cup 8}) = 400 - 160$

$= 240$

Q At a construction site, George is the incharge of hiring skilled workers for the project. Out of 80 candidates that he interviewed, he found that

45 of them are painters
50 of them are electricians
50 of them are plumbers
15 of them are skilled in all 3 areas

All of them had skills in atleast one of these areas

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which of the following is NOT TRUE?

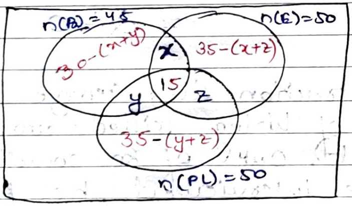
a) No. of candidates with atleast two skills = 50

b) No. of candidates with atmost two skills = 55

c) No. of candidates with exactly two skills = 35

d) No. of candidates with only one of the three skills = 30.

2:22:30



~~A opt~~

a) $30-(x+y)+35-(x+z)+35-(y+z)+(x+y+z)+15 = 80$

$115-(x+y+z) = 80$

$x+y+z = 35$

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a) $x+y+z+15 = 50$ (TRUE)

c) $x+y+z = 35$ (TRUE)

d) $30-(x+y)+35-(x+z)+35-(y+z) = 30$ (TRUE)

b) $[30-(x+y)+35-(x+z)+35-(y+z)] + (x+y+z) \neq 55$ (FALSE)

2:27

Q Let A, B, C, D are sets so that n(A) = 42, n(B) = 36, n(C) = 28, n(D) = 24, Intersection of any two ~~element~~ of these 4 sets contain 12 elements, intersection of any three of these three contain 8 elements and $n(A \cap B \cap C \cap D) = 4$, then number of elements in $(A \cup B \cup C \cup D)$ is ——.

$n(A \cup B \cup C \cup D)$ Use alternative sign

$^4C_1 = 4$

$\rightarrow n(A)+n(B)+n(C)+n(D)$

$^4C_2 = 6$

$\rightarrow -n(A\cap B)-n(A\cap C)-n(A\cap D)-n(B\cap C)-n(B\cap D)-n(C\cap D)$

$^4C_3 = 4$

$\rightarrow +n(A\cap B\cap C)+n(A\cap B\cap D)+n(A\cap C\cap D)+n(B\cap C\cap D)$

$\xrightarrow{} {}^4C_4 = 1$

$-n(A \cap B \cap C \cap D)$

ATQ:

$n(A \cup B \cup C \cup D)$

$= \underbrace{42 + 36 + 28 + 24}_{^4C_3 = 4} - \underset{^4C_2}{6}(12) + \underset{^4C_1}{4}(8) - 4$

$= 86$

GATE-04

Q In a class of 200 students, 125 students have taken Programming language course, 85 students have taken Data structure course, 65 students have taken Comp Organization

n(Prog Lang ∩ Data Struc) = 50
n(Prog Lang ∩ Comp Organisation) = 35
n(Data struc ∩ Comp. Org.) = 30
n(Prog. Lang ∩ Data struc ∩ Comp Org) = 15

How many students have not taken any of the three courses?

a) 15 b) 20 c) 25 d) 35

$= \text{~~50 + 35 + 30~~}$

$= 125 + 80 + 65 - 50 - 35 - 30 + 15$

$= 175$

$n(\overline{A \cup B \cup C}) = 200 - 175 = 25$ Ans

GATE-98

Q Total = 28
E = 18, H = 13, K = 22
$E \cap H = 9$, $H \cap K = 11$, $E \cap K = 13$
$n(E \cap H \cap K) = ?$

$28 = 18 + 13 + 22 - 9 - 11 - 13 + x$

$28 = 22 + x$
$x = 6$

Q What is the cardinality of the set of integer X defined below?

$X = \{ n \mid 1 \leq n \leq 123,$ n is not divisible by either 2, 3 or 5 $\}$

a) 28
b) 33
c) 37
d) 44

Lecture 10B

Derangement

None of the object arranged in its natural or original position.

Original Position :- 1st 2nd 3rd

a b c

1st 2nd 3rd

a b c

a c b

b a c

→ b c a

→ c a b

c b a

$\} D_3 = 2$

1st 2nd

a b

1st 2nd

a b

b a

$\} D_2 = 1$,

1st

a , $D_1 = 0$

$D_1 = 0$, $D_2 = 1$

$$D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots + \frac{(-1)^n}{n!}\right]$$

$$D_3 = 3!\left[\frac{1}{2!} - \frac{1}{3!}\right]$$

$$D_3 = 3!\left[\frac{3-1}{6}\right]$$

$$D_3 = 2$$

learn them

$$\boxed{D_4 = 9, \quad D_5 = 44}$$

Else

$$\boxed{D_n = (n-1)\left[D_{n-1} + D_{n-2}\right] \text{ for } n \geq 3}$$

Q How many 1-1 functions are possible on a set with 6 elements so that no element is mapped to itself?

$$\therefore D_6 = 6!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right]$$

$D_6 = 265$ Ans

Q In how many ways can we put 5 letters L_1, L_2, L_3, L_4, L_5, in 5 envelopes e_1, e_2, e_3, e_4, e_5 (at 1 letter per envelope) so that

i) No letter is correctly placed
ii) Atleast 1 letter is correctly placed
iii) exactly 2 letters are correctly placed
iv) atmost 1 letter is correctly placed
v) atleast 1 letter is wrongly placed
vi) exactly 1 letter is wrongly placed?

i) $D_5 = 44$

ii) $5! - D_5 = 120 - 44 = 76$

iii) $^5C_2 \times \textcircled{1} \times D_3 = 10 \times 2 = 20$

↳ 2 letters in correct place in only 1 way.

iv) $D_5 + (^5C_1 \times 1 \times D_4) = 44 + 5 \times 9 = 89$

5!

v) $120 - \textcircled{1} = 119$

↳ Ways in which every letter in correct place

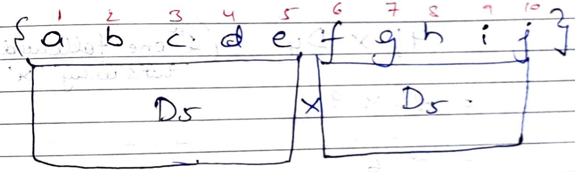
vi) 0 → Because if one letter is in wrong place then the corresponding letter to which the first letter goes also go to some another place that is wrong place.

So exactly 1 letter wrongly placed not possible.

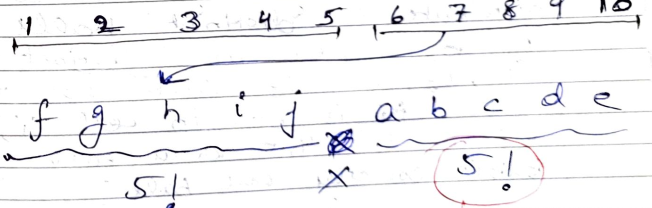
Q Number of derangements possible for the sequence {a, b, c, d, e, f, g, h, i, j} so that

i) The first 5 letters of the sequence are in first 5 places

ii) None of the first 5 letters of the sequence are in first 5 places?

i) 

44 × 44

ii) 

Every arrangement is derangement as the already are in wrong place.

Q Suppose 4 different books are distributed among 4 students (@ 1 book per student). Further suppose, the books were returned by the students and again distributed among the students later on. In how many ways this can be done so that no student can take the same book twice?

Sol $4! \times D_4$ (1st one followed by 2nd that's why '×')

$= 24 \times 9$

$= 216$

GATE-04

Q In how many ways can we distribute 5 distinct balls, $B_1, B_2, \ldots, B_5$ in 5 distinct cells, $C_1, C_2, \ldots, C_5$ such that ~~#~~ Ball B_i is not in cell C_i; $\forall i = 1, 2, 3, 4, 5$ and each cell contains exactly one ball?

a) 44 b) 96 c) 120 d) 3125

Sol $D_5 = 44$

∴ A option

Q A party was attended by n guest. When the guest arrived they left their Hat in the coatroom. After the party ended, there was power failure ~~and~~ so each guest took the hat at random. When they out on street they were amused that ~~no one get the~~ none of them get his hat back. In how many different ways could that happen?

a) $\sum_{i=0}^{n} (-1)^{i} \frac{n!}{i!}$ b) $\sum_{i=0} (-1)^{i-1} \frac{n!}{i!}$

c) $\sum_{i=0}^{n} (-1)^{i} \frac{(n+1)!}{i!}$ d) $\sum_{i=0}^{n} (-1)^{i-1} \frac{(n+1)!}{i!}$

Sol $D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + \frac{(-1)^n}{n!}\right]$

∴ Option A is correct

* Derangement formula same as no. of onto functions.

Generating function

Transforming problems about sequence into problems of functions

$\langle a_0 \; a_1 \; a_2 \; \ldots \; a_i \; \ldots \rangle$

$\cong a_0x^0 + a_1x^1 + a_2x^2 + \ldots + a_ix^i + \ldots$

Rule:- 'a_i' is always coefficient of x^i

* Generating functions for the following sequencies:

i) $\langle \overset{x^0}{1} \; \overset{x^1}{1} \; \overset{x^2}{1} \; 1 \; 1 \ldots \rangle \cong x^0 + x^1 + x^2 + \ldots$

$\cong 1 + x + x^2 + x^3 + \ldots$

ii) $\langle 1 \; 2 \; 3 \; 4 \; 5 \ldots \rangle \cong x^0 + 2 \cdot x^1 + 3x^2 + 4x^3 + \ldots$

$\cong 1 + 2x + 3x^2 + 4x^3 + \ldots$

iii) $\langle 1 \; -1 \; 1 \; -1 \; 1 \; -1 \ldots \rangle$

$\cong x^0 - x^1 + x^2 - x^3 + x^4 - x^5 \ldots$

$\cong 1 - x + x^2 - x^3 + \ldots$

(Without Repeatition)

1.) No. of ways of choosing 'k' objects from $\{a_1\}$
$= 1 + x$

$$\begin{array}{lccccl} & 0 & 1 & 2 & 3 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{No. of ways} = & \langle 1 & \boxed{1} & 0 & 0 & \ldots \rangle \cong 1 \cdot x^0 + 1 \cdot x \\ & \uparrow & \uparrow & \uparrow & & \cong 1 + x \\ & x^0 & x^1 & x^2 & \ldots & \end{array}$$

no. of ways of selecting 2 object from set of 1 element

No. of ways of selecting 1 object from set of 1 element

2.) G.f for no. of ways for choosing 'k' objects from $\{a_2\}$ $= 1 + x$

Generating function of $(a_1)(a_2)$

3) G.f for no. of ways for choosing 'k' objects from $\{a_1, a_2\}$ $= (1+x)(1+x)$
$= (1+x)^2$

$$\begin{array}{lcccccl} & 0 & 1 & 2 & 3 & 4 & \ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{ways}: & \langle 1 & 2 & 1 & 0 & 0 & \ldots \rangle = 1 \cdot x^0 + 2 \cdot x^1 + 1 \cdot x^2 \\ & \uparrow & \uparrow & \uparrow & & & \\ & x^0 & x^1 & x^2 & & & \end{array}$$

$= 1 + 2x + x^2$

$= (1+x)^2$

4) G.F for no. of ways of choosing 'k' objects $\{a_1, a_2, a_3\}$ $= (1+x)(1+x)(1+x)$

0 1 2 3 4

ways : < 1 3 3 1 0 ---->

no. of ways of selecting 1 object from set

x^0 x^1 x^2 x^3

$= 1 + \boxed{3}x^1 + 3x^2 + x^3$

$= (1+x)^3$

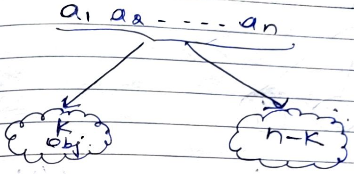
* G.F for no. of ways of choosing 'k' objects from $\{a_1, a_2, \ldots, a_n\}$

$= (1+x)^{n}$ n → no. of elements in the set

Note: No. of ways for choosing 'k' objects of $\{a_1, a_2, \ldots, a_n\}$ without Repeatition

= Coefficient of x^k in $(1+x)^n$

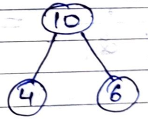
$= {}^{n}C_{K}$



$${}^{n}C_{K} = \frac{n!}{K!\,(n-K)!}$$

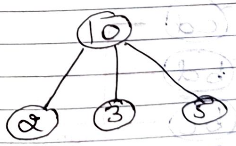
Pb How many ways we can divide 10 objects into two groups such that one group contain 4 objects.

${}^{10}C_4 \times 1$

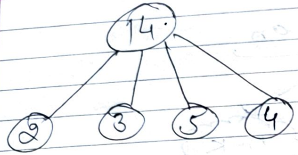


$\frac{10!}{4!\,6!}$

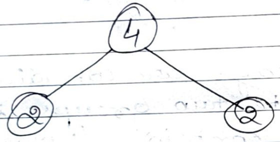
Same Ans.



${}^{10}C_2 \times {}^{8}C_3 \times {}^{5}C_5$ or $\frac{10!}{2!\,3!\,5!}$



$${}^{14}C_2 \times {}^{12}C_3 \times {}^{9}C_5 \times {}^{4}C_4 \text{ or } \frac{14!}{2!\,3!\,5!\,4!}$$

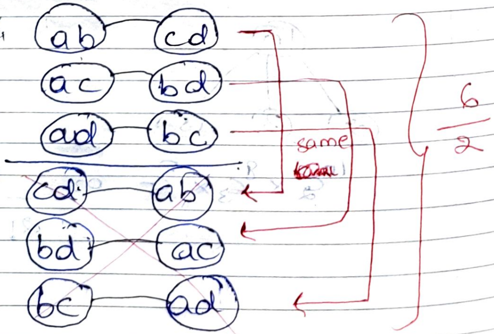


$$\frac{4!}{2!\;2!} = \frac{4\times3\times2\times1}{2!\times2!}$$

$$= 6$$

a b c d

Eg



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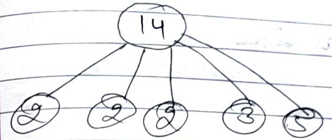
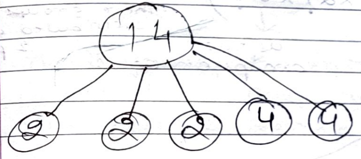


$$\frac{14!}{2!\,2!\,2!\;3!\,5!\;3!}$$

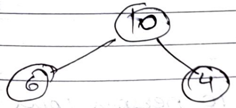
three groups are of same size

*



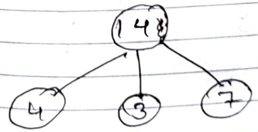
$$\frac{14!}{2!\,2!\,2!\;4!\,4!\;2!\,3!}$$

Division & Distribution Concept



A	B	
6	4	2!
4	6	

$$\left(\frac{10!}{6!\times4!}\right)\times 2!$$



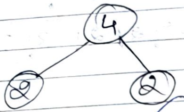
A	B	C	
4	3	7	3!
–	–	–	
–	–	–	
–	–	–	

$$\left(\frac{14!}{4!\;3!\;7!}\right)\times 3!$$

No. of ways in which these groups are distributed among 3 people.

★ Groups of equal sizes

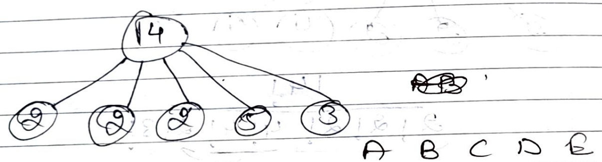
A B



$$\left(\frac{4!}{2!\times 2!\times 2!}\right)\times 2!$$

↳ ways in which there groups are distributed among 2 person.

↓ as two groups are equal



$$\left(\frac{14!}{2!\,2!\,2!\,5!\,3!\,3!}\right)\times 5!$$

Q How many ~~to~~ ways 10 person can be divided into 3 teams so that

Team 1 contains 3 members and
" 2 " 2 " "
" 3 " 5 " " ?

$$\frac{10!}{3!\,2!\,5!}$$

Q How many ways 10 person can be divided into 5 teams of 2 each.

A

$$\frac{10!}{2!\,2!\,2!\,2!\,2!\times 5!}$$

↓ 5 groups of equal size.

Q How many binary sequence of length 10 are possible with exactly 3 zeroes?

_ _ _ _ _ _ _ _ _ _

Method 1

$$\frac{10!}{3!\,7!}$$

Method 2

$${}^{10}C_3\times 1$$

Method 3

~~1 or 0~~

1 (or) 0

↓ ↓

x y

~~GF~~ Generating f^n = $(x+y)^{10}$

As we want 3 zeroes

∴ coeff of y^3x^7 in the expansion of $(x+y)^{10}$

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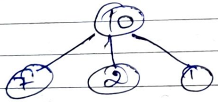
Q How many binary sequences of length 10 are possible with exactly 4 zeroes and no two zeroes are consecutive?

$^7C_4 \times 1$

Q Number of strings possible with seven 0's, two 1's and one 2 is ______.



_ _ _ _ _ _ _ _ _ _

Method 1

$$\frac{10!}{7!\,2!\,1!}$$

Method 2

$$^{10}C_7 \times {}^3C_2 \times {}^1C_1$$

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With Repeatition

* For no. of ways for choosing 'k' objects from $\{a_1\} = (1-x)^{-1}$

0 1 2 3

a_1 a_1a_1 $a_1a_1a_1$

ways: < 1 1 1 1 ......>

$= 1 + x + x^2 + x^3 + \dots$

$= \frac{1}{1-x}$ (∵ here $a=1$, $r=x$)

* For no. of ways for choosing 'k' objects $\{a_2\} = (1-x)^{-1}$.

* For no. of ways for choosing 'k' obj $\{a_1, a_2\} = (1-x)^{-1}(1-x)^{-1}$

$= (1-x)^{-2}$

0 1 2 3 4'

Ways: < 1 2 3 ......> $= 1 + 2x + 3x^2 + \dots$

$\uparrow$ $\uparrow$ $\uparrow$

x^0 x^1 x^2

$= (1-x)^{-2}$

* for no. of ways of choosing 'k' object from $\{a_1, a_2, a_3\}$

$$= (1-x)^{-1} (1-x)^{-1} (1-x)^{-1}$$

$$= (1-x)^{-3}$$

* G.F for no. of ways of choosing 'k' obj of $\{a_1, a_2, ----, a_n\}$ with Repeatition $= (1-x)^{-n}$

Note: No. of ways for choosing 'k' objects of $\{a_1, a_2, ---, a_n\}$ with repeatition

= Coefficient of x^k in the expansion of $(1-x)^{-n}$

$$= {}^{n+k-1}C_k$$

Proof of previous formula

Let $f(x) = (1-x)^{-n}$

Taylor's Theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^k}{k!} f^k(0) + -----$$

Here coefficient of $x^k = \frac{f^k(0)}{k!}$

$$f(x) = (1-x)^{-n}$$

$$f'(x) = -n(1-x)^{-n-1} \frac{d}{dx}(1-x)$$

$$\left(\because \frac{d}{dx} x^n = n x^{n-1}\right)$$

$$f'(x) = n(1-x)^{-(n+1)}$$

$$f''(x) = -n(n+1)(1-x)^{-n+2} \frac{d}{dx}(1-x)$$

$$f''(x) = n(n+1)(1-x)^{-(n+2)}$$

$$\vdots$$

$$f^k(x) = n(n+1)(n+2)\dots(n+k-1)(1-x)^{-(n+k)}$$

Now,

$$\frac{f^{K}(0)}{K!} = \frac{n(n+1)(n+2)---(n+K-1)}{K!}$$

$$= \frac{1\cdot2\cdot3\cdots(n-1)n(n+1)(n+2)---(n+K-1)}{1\cdot2\cdot3\cdots(n-1) \quad K!}$$

$$= \frac{(n+K-1)!}{(n-1)!\ K!}$$

$$= {}^{(n+K-1)}C_K$$ is coeff. of x^K in expansion of $(1-x)^{-n}$

2:27:00

✡ No. of ways choosing 'k' object of $\{a_1, a_2, ----, a_n\}$ with repeatations allowed

$$= {}^{(n+K-1)}C_K$$

① No. of ways distributing 'K' similar objects into 'n' boxes

$$= {}^{n+K-1}C_K$$

② No. of non-negative integral solutions for the following equation

$$x_1 + x_2 + ---- + x_n = K$$

$$\rightarrow {}^{(K+n-1)}C_{n-1} \cong {}^{(n+K-1)}C_K$$

Non-negative integral solution

① $x_1 + x_2 + x_3 = 10$

$$^{10+3-1}C_{3-1} \cong {}^{12}C_2$$

② $x_1 + x_2 + x_3 + x_4 = 15$

$$^{(15+4-1)}C_{4-1} = {}^{18}C_3$$

③ $x_1 + x_2 + x_3 + x_4 + x_5 = 20$

$$^{20+4}C_4 = {}^{24}C_4$$

○ ○ ○ ○ ○ ○ ○ ○ ○ ○ + + (10 similar balls)

& in between or at the end we have to insert these '+' sign's

$\therefore$ $^{10+2}C_2$

o o + o o . o o + o o o o o

$x_1 = 2$ $x_2 = 4$ $x_3 = 4$

$$\underbrace{o\ o\ o\ o\ o}_{x_1 = 5} + \underbrace{o\ o\ o\ o\ o}_{x_2 = 5} + \quad x_3 = 0$$

Like this we have to insert 2 '+' sign as there are 3 variables.

If there are 4 variables then we have to use 3 '+' sign.

General

$$x_1 + x_2 + \dots + x_n = K$$

$$^{K + (n-1)}C_{n-1} = {}^{n+k-1}C_k$$

K → sum of no. of similar balls

$(n-1)$ → no. of '+' sign

Modal Questions

Model 1 :

No. of non negative integer solutions for $x_1 + x_2 + x_3 = 10$

$$^{10+2}C_2 = {}^{12}C_2$$

Model 2 :-

No. of non-negative integer solutions

$$x_1 + x_2 + x_3 = 15, \quad x_1 \geq 2, \ x_2 \geq 3, \ x_3 \geq 4$$

Method 1

After fullfilling the required condition the new equation becomes

$$x_1 + x_2 + x_3 = 6 \quad \begin{pmatrix} 4+3+2 = 9 \\ 15-9 = 6 \end{pmatrix}$$

↳ $x_1, x_2, x_3 \geq 0$

$$\therefore {}^{(6+2)}C_2 = {}^{8}C_2$$

Method 2

Let $x_1 = y_1 + 2$, $y_1 \geq 0$

$x_2 = y_2 + 3$, $y_2 \geq 0$

$x_3 = y_3 + 4$, $y_3 \geq 0$

$$x_1 + x_2 + x_3 = 15$$

$$(y_1 + 2) + (y_2 + 3) + (y_3 + 4) = 15$$

$$y_1 + y_2 + y_3 = 6$$

$$^{(6+2)}C_2 = {}^{8}C_2$$

Model 3

$x_1+x_2+x_3 \le 15$, where $x_1, x_2, x_3 \ge 0$

$x_1+x_2+x_3+t=15$,

~~where~~ where, $t = 15-(x_1+x_2+x_3)$

$\therefore t \ge 0$

$\therefore {}^{15+3}C_3 = {}^{18}C_3$

Model 4

$x_1+x_2+x_3 < 15$, where $x_1\ x_2\ x_3 \ge 0$

$x_1+x_2+x_3+t=15$,

$t = 15-(x_1+x_2+x_3)$

$\therefore t > 0$

$\therefore t \ge 1$

After fulfilling the condition

$x_1+x_2+x_3+t = 14$

$= {}^{14+3}C_3$

$= {}^{17}C_3$

OR

$x_1+x_2+x_3 < 15$

$\therefore x_1+x_2+x_3 \le 14$

$\Rightarrow x_1+x_2+x_3+t_1 = 14$

where $t = 14-(x_1+x_2+x_3)$

$\therefore t \ge 0$

${}^{14+3}C_3 = {}^{17}C_3$

Q Number of non negative solutions to the following inequality $12 \le w+x+y+z \le 14$ is ___.

Method 1 $w+x+y+z=12$

$^{(12+3)}C_3 = {}^{15}C_3$

$w+x+y+z=13$

$^{(13+3)}C_3 = {}^{16}C_3$

$w+x+y+z=14$

${}^{17}C_3$

${}^{15}C_3 + {}^{16}C_3 + {}^{17}C_3$

Method 2

$12 \leq w+x+y+z \leq 14$

$(0 \leq w+x+y+z \leq 14) - (0 \leq w+x+y+z \leq 11)$

$w+x+y+z+t_1 = 14 \, , \quad w+x+y+z+t_2 = 11$

$^{14+4}C_4 \; - \; ^{11+4}C_4$

$= \; ^{18}C_4 - \; ^{15}C_4$ Ans

Q Number of integral solutions to the equations

$x_1 + x_2 + x_3 = 8$

where $x_1 \geq 3$, $x_2 \geq -2$ and $x_3 \geq 4$ is ________.

$x_1 = y_1 + 3$

$x_2 = y_2 - 2$

$x_3 = y_3 + 4$

$\therefore \; y_1, y_2, y_3 \geq 0$

$x_1 + x_2 + x_3 = 8$

$(y_1 + 3) + (y_2 - 2) + (y_3 + 4) = 8$

$y_1 + y_2 + y_3 = 3$

$= \; ^{(3+2)}C_2$

$= \; ^5C_2$

$= 10$

Q No. of ways we can distribute 15 similar balls among three distinct boxes so that no box will contain more than 7 balls is ________.

a) 24 b) 28

c) 32 d) 36

1:05:15

Ans already calculated by me (swiray) :)

$1 \leq x_1 \leq 7$; ~~$y_1 \geq 0$~~

$1 \leq x_2 \leq 7$;

$1 \leq x_3 \leq 7$;

$x_1 = 7 - y_1 , \quad y_1 \geq 0$

$x_2 = 7 - y_2 , \quad y_2 \geq 0$

$x_3 = 7 - y_3 , \quad y_3 \geq 0$

$x_1 + x_2 + x_3 = 15$

$(7-y_1) + (7-y_2) + (7-y_3) = 15$

$y_1 + y_2 + y_3 = 6$

${}^{6+2}C_2 = 28$

<u>Method 2</u> (By Generating Function)

$x_1 + x_2 + x_3 = 15$

$\langle 0\,1\,1\,1\,1\,1\,1\,1\,0\,0 \dashrightarrow$

$\langle 0\,1\,1\,1\,1\,1\,1\,1\,0\,0 \dashrightarrow$

$x^0 \quad x^1 \quad x^2 \quad x^3 \quad x^4 \quad x^5 \quad x^6 \quad x^7$

$\langle 0\,1\,1\,1\,1\,1\,1\,1\,0\,0 \dashrightarrow$

For x, the G.F is

$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$

For $\{x_1, x_2, x_3\}$, the generating function is

$(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7)^3$

$x^3 [1 + x + x^2 + x^3 + x^4 + x^5 + x^6]^3$

find coeff of x^{15} in the above

as $(x_1 + x_2 + x_3 = 15)$

$= x^3 \left[\frac{1-x^7}{1-x}\right]^3$

$S_n = \frac{a(1-r^n)}{1-r}$

$a = 1, \; r = x$

$S_7 = \frac{1(1-x^7)}{1-x}$

$= x^3 (1-x^7)^3 (1-x)^{-3}$

$= x^3 (1 - x^{21} - 3x^7 + 3x^{14})(1-x)^{-3}$

$= (x^3 - 3x^{10})(1-x)^{-3}$

Multiplying x^{21} & x^3, x^{14} & x^3 we will not get x^{15} $\boxed{n = 3}$

$= (x^3 - 3x^{10}) \sum_{k=0}^{\infty} {}^{(3+k-1)}C_k \, x^k$

$= (x^3 - 3x^{10}) \left({}^{3+12-1}C_{12} \, x^{12} + {}^{3+5-1}C_5 \, x^5 \right)$

$= {}^{14}C_{12} - 3 \times {}^{7}C_5$

$= 91 - 63 = 28$ Ans

GATE -16 - set 1

Q The coefficient of x^{12} in $(x^3+x^4+x^5+x^6+ ---)^3$ is

$= (x^3+x^4+x^5+x^6+ ---)^3$

$= x^9 [1+x+x^2+x^3+ --]^3$

$= x^9 \left[\frac{1}{1-x}\right]^3$ $\left[S_\infty = \frac{a}{1-r} \text{ here } a = 1,\ r = x \right]$

$= x^9 (1-x)^{-3}$

We know that

$$(1-x)^{-n} = \sum_{k=0}^{\infty} {}^{(n+k-1)}C_k\, x^k$$

$$x^9(1-x)^{-3} = x^9 \left[\sum_{k=0}^{\infty} {}^{(3+k-1)}C_k\, x^k\right]$$

$$= x^9 \left[\sum_{k=0}^{\infty} {}^{(2+k)}C_k\, x^k\right]$$

for $k=3$, we get x^{12}

$= x^9 \; {}^{(2+3)}C_3 \; x^{12}$

$= {}^5C_3\, x^{12}$

$= 10 x^{12}$

∴ Coeff of x^{12} = 10

GATE -17 set 2

Q If the ordinary generating function of a sequence $\{a\}_{n=0}^{\infty}$ is $\frac{1+z}{(1-z)^3}$, then $\underset{\text{Coeff of } z^3}{a_3} - \underset{\text{Coeff of } z^0}{a_0}$ is equal to

$$\frac{1+z}{(1-z)^3} = (1+z)(1-z)^{-3}$$

Taylor's Expansion

$$= (1+z)\left[\sum_{k=0}^{\infty} {}^{(3+k-1)}C_k\, z^k\right]$$

To get a_3, a_0 we have to find coeff of z^3, z^0 respectively

$$= (1+z)\left[{}^2C_0 z^0 + {}^3C_1 z^1 + {}^4C_2 z^2 + {}^5C_3 z^3 \ldots\right]$$

$$= (1+z)\left[\overset{a_0}{1} + 3z + 6z^2 + \overset{a_3}{10z^3} + ---\right]$$

a_3

∴ $a_3 = 10 + 6$

$a_0 = 1$

$a_3 - a_0 = 16 - 1 = 15$

25/4/21 Lecture 12 A

Recurrence Relations

- Discrete Maths
- Algorithms
- Data Structure
    - Stack
        - Recursion
- Compiler Design

} All contain Recurrence Relation

Definition

Let $\{a_0, a_1, a_2, \ldots, a_n, \ldots\}$ be a sequence of real numbers. A formula that relates (present term) with one (or) more of the preceeding terms is called a Recurrence Relation.

Ex1: Let S_n denotes the sum of the first 'n' positive integers then

$$S_n = S_{n-1} + n$$

$$S_n = \boxed{1+2+3+\cdots+n-1} + n$$

$$S_n = S_{n-1} + n$$

Ex: 2: If 'd' is the real number, then the n^{th} term of an Airthmetic Progression with common difference "d" satisfies

$$t_n = t_{n-1} + d$$

Ex-3 Fibonacci Series

0 1 1 2 3 5 8 13 21 ...

$$f_n = f_{n-1} + f_{n-2}$$

$$\underbrace{\underline{0/1}\quad \underline{0/1}\quad \underline{0/1}\quad \underline{-\,-}\quad \underline{-\,-}\quad \underline{0/1}\quad \underline{0/1}}_{n}$$

Number of

Binary sequence of length 'n' $= 2^n$
$\{0,1\}$.

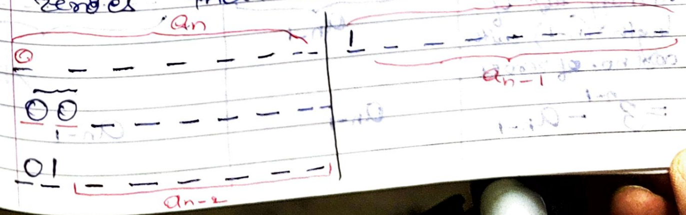
Ternary sequence of length 'n' $= 3^n$
$\{0,1,2\}$.

$$\overset{0|1|2}{x_1}\quad \overset{0|1|2}{x_2}\ \cdots\cdots\ \overset{0|1|2}{x_n}$$

Quaternary sequence of length 'n' $= 4^n$
$\{0,1,2,3\}$

Formation of Recurrence Relations

If a_n = Number of binary sequences of length 'n' with no consecutive zeroes then recurrence for a_n is



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$$a_n = a_{n-1} + a_{n-2}$$

Q.2 If a_n = No. of ternary sequences of length 'n' with even no. of zeroes then the recurrence relation for a_n is ______

$$\overset{0|1|2}{x_1}\ \ \overset{0|1|2}{x_2}\ \ \overset{0|1|2}{x_3} \ ----\ \overset{0|1|2}{x_n} = 3^n$$

No. of ternary sequence of length 'n' = 3^n

↓

$\underbrace{\text{No. of ternary seq. of even no. of zeroes}}_{a_n}$ + No. of ternary seq of odd no. of zeroes $= 3^n$

a_n + No. of Ternary seq of odd no. of zeroes $= 3^n$

No. of Ternary seq of length 'n' of odd no. of zeroes $= 3^n - a_n$

$$\underline{0}\ \_\ \_\ \_\ \_ \;\Big|\; \underline{1}\ \underbrace{\_\ \_\ \_\ \_\ \_}_{a_{n-1}} \;\Big|\; \underline{2}\ \underbrace{\_\ \_\ \_\ \_}$$

Ternary seq. of length (n-1) with odd no. of zeroes $= 3^{n-1} - a_{n-1}$ | a_{n-1} | a_{n-1}

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$$\therefore a_n = 3^{n-1} - a_{n-1} + 2a_{n-1}$$
$$= 3^{n-1} + a_{n-1}$$

$$\underline{0}\ \underbrace{\_\ \_\ \_\ \_}_{(3^{n-1} - a_{n-1})} \;\Big|\; \overset{1/2}{\_}\ \_\ \_\ \_ \quad {}^2C_1 \times a_{n-1}$$

$$\therefore a_n = 2a_{n-1} + 3^{n-1} - a_{n-1}$$
$$a_n = a_{n-1} + 3^{n-1}$$

Q The recurrence relation for the number of n-digit quaternary sequences that have an even no. of zeroes (quaternary sequences use only 0, 1, 2, 3 for digits) is ______.

a) $a_n = a_{n-1} + 4^{n-1}$

b) $a_n = 3a_{n-1} + 4^{n-1}$

c) $a_n = 2a_{n-1} + 4^{n-1}$

(d) $a_n = a_{n-1} + 4^{n-1}$

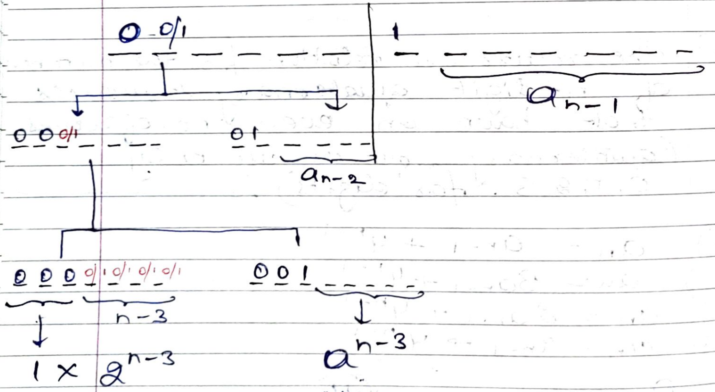
$$\underline{0}\ \underbrace{\_\ \_\ \_\ \_\ \_}_{4^{n-1} - a_{n-1}} \;\Big|\; \overset{1|2|3}{\_}\ \underbrace{\_\ \_\ \_\ \_}_{a_{n-1}} \quad {}^3C_1$$

∴ $a_n = 4^{n-1} - a_{n-1} + 3a_{n-1}$

$= 2a_{n-1} + 4^{n-1}$

Q If a_n = number of bit strings of length 'n' with 3 consecutive zeroes, then recurrence relation for a_n is ______.

a) $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$
b) $a_n = a_{n-1} + a_{n-2} + a_{n-3}$
c) $a_n = a_{n-1} + a_{n-2} + a_{n-3} - 2^{n-3}$
d) $a_n = (n-4)\,a_{n-3}$



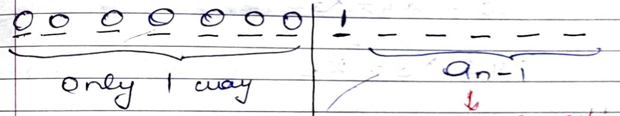
∴ $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

1:07:30

26/4/21

Q2 Let a_n be the number of bit strings of length n s.t. that do not contain ~~within~~ 0/1 01. The recurrence relation for a_n is ______.

a) $a_n = a_{n-1} + 1$
b) $a_n = a_{n-1} + 2^{n-2}$
c) $a_n = 3a_{n-2} + 2^{n-2}$
d) $a_n = a_{n-1} + a_{n-2}$



because if 1 comes in between then 01 condition satisfy which we do not want

we are expecting condition that 01 is not contain...

$a_n = a_{n-1} + 1$

Option A → Correct Answer

GATE - 08

Q Let x_n denote the number of binary strings of length n that contains no consecutive 0's.
The value of x_5 is

a) 5 b) 7
c) 8 d) 13

$x_n = x_{n-1} + x_{n-2}$

$x_5 = x_4 + x_3$

$x_4 = x_3 + x_2$

$x_3 = x_2 + x_1$

n	Possible	
1	0 1	→ $x_1 = 2$
2	~~0 0~~ 0 1 1 0 1 1	$x_2 = 3$
3	~~0 0 0~~ ~~0 0 1~~ 0 1 0 0 1 1 ~~1 0 0~~ 1 0 1 1 1 0 1 1 1	$x_3 = 5$

$x_3 = 5$; $x_2 = 3$

$x_4 = x_3 + x_2$

$x_4 = 5 + 3$

$x_4 = 8$

$x_5 = x_4 + x_3$

$x_5 = 8 + 5$

$x_5 = 13$

GATE - 15 - set 1

Q Let a_n represent the number of bit strings of length n containing two consecutive 1's. What is the recurrence relation for a_n?

a) $a_{n-2} + a_{n-1} + 2^{n-2}$
b) $a_{n-2} + 2a_{n-1} + 2^{n-2}$
c) $2a_{n-2} + a_{n-1} + 2^{n-2}$
d) $2a_{n-2} + 2a_{n-1} + 2^{n-2}$

$\underbrace{0 \_ \_ \_ \_}_{a_{n-1}}$ | 1 0/1 _ _ _ _ _

$1\ 0\ \underbrace{\_ \_ \_ \_ \_}_{a_{n-2}}$ $1\ 1\ \underbrace{0/1\ 0/1\ 0/1\ \_ \_ \_}_{2^{n-2}}$

$a_{n-1} + a_{n-2} + 2^{n-2}$

a

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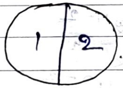
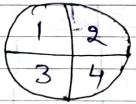
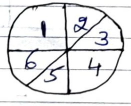
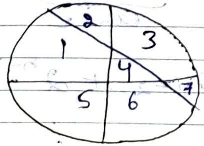
Q The recurrence relation for the maximum number of pieces of a pizza made by n straight cuts is

a) $f(n) = f(n-1) + n$
b) $f(n) = f(n-1) + 2n$
c) $f(n) = f(n-1) + n^2$
d) $f(n) = f(n-1) + (n-1)$

Let f(n) = Maximum no. of pieces of pizza made by 'n' straight cuts.

n	Pizza	f(n)
1		$\therefore f(1) = 2$
2		$\therefore f(2) = 4$
3	 	$f(3) = 7 = 4+3$ $f(3) = f(2) + 3$

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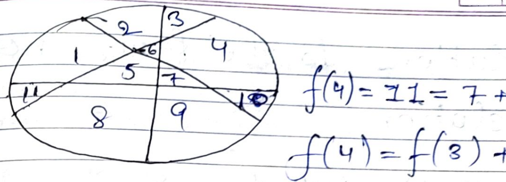
4) 

$f(4) = 11 = 7 + 4$

$f(4) = f(3) + 4$

In General,

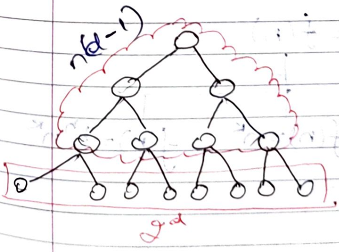
$$\therefore f(n) = f(n-1) + n$$

Q The recurrence relation for the maximum number of nodes in a binary tree of depth d is ______.

a) $n(d) = d(n-1) + 2^d$
b) $n(d) = n(d-1) + 2^d$
c) $n(d) = n(d-1) + 2^n$
d) $n(d) = d(n-1) + 2^n$

n(d) → Max. no. of nodes of a binary tree with depth 'd'.

Let $n(d) = n(d-1) + 2^d$



Depth (d)	No. of nodes
0	$1 = 2^0$
1.	$2 = 2^1$
2	$4 = 2^{②}$
3	$8 = 2^{③}$

$$\therefore n(d) = \underset{\text{Previous all}}{n(d-1)} + \underset{\text{current depth}}{2^d}$$

Solutions to the Recurrence Relation

Dmaths

1.) Substitution Method

2.) Method of characteristic roots

3.) Method of undetermined coefficients

4.) Master Theorem (Divide & Conquer) } → Algorithms

5.) Recursion Tree.

Types

- Homogeneous ⇒ $\phi(n) = 0$
  - Ex. $a_n - 2a_{n-1} = 0$
- Non-Homogeneous: $\phi(n) = f(n)$
  - $f(n) = b^n$
  - $f(n) = n^k$
  - $f(n) = b^n n^k$

GATE-04

Q The recurrence equation

$$T(1) = 1$$

and $T(n) = 2T(n-1) + n \quad (n \geq 2)$

evaluates to

a) $2^{n+1} - n - 2$ b) $2^n - n$

c) $2^{n+1} - 2n - 2$ d) $2^n + n$

Substitution Method

$$T(1) = 1$$
$$T(n) = 2T(n-1) + n$$

Let $n = 2$

$$T(2) = 2T(1) + 2$$
$$= 2 \times 1 + 2$$
$$T(2) = 4$$

Substitute $n=2$ in given 4 option whichever produces 4; that will be the correct Answer

A → correct option

GATE-02

Q The solution to the recurrence equation

$T(2^k) = 3T(2^{k-1}) + 1$,

$T(1) = 1$ is :

a) 2^k b) $(3^{k+1}-1)/2$

c) $3^{\log_2 k}$ d) $2^{\log_3 k}$

If K=1

$T(2) = 3T(2^0) + 1$

$= 3T(1) + 1$

$= 3(1) + 1$

$= 4$

∴ In option B

By putting K=1

$(3^{k+1}-1)/2 = 4$

∴ B - Correct Option

Q The solution of $a_n = a_{n-1} + \frac{1}{n(n+1)}$ where $a_0 = 1$ is

a) $\frac{2n+1}{n+1}$, b) $\frac{2n-1}{n+1}$

c) $\frac{2n+1}{n-1}$ d) $\frac{2n-1}{n-1}$

$a_1 = a_0 + \frac{1}{1 \cdot 2}$

$a_1 = 1 + \frac{1}{2}$

$a_1 = \frac{3}{2}$

Option A :

$n = 1$, $\frac{2+1}{1+1} = \frac{3}{2}$ Au

Option B, C, D not providing $\frac{3}{2}$ by putting $n=1$.

Long method

$n_1 = 1$

$a_1 = a_0 + \frac{1}{1 \cdot 2} = 1 + \frac{1}{1 \cdot 2} = 1 + \left(\frac{1}{1} - \frac{1}{2}\right)$

~~$n_2 = 2$~~

~~$a_2 = a_1 + \frac{1}{2 \cdot 3}$~~

$n = 2$

$a_2 = a_1 + \frac{1}{2 \cdot 3} = 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$

$a_2 = 1 + \left(\frac{1}{1} - \frac{1}{3}\right)$

$n = 3$

$a_3 = a_2 + \frac{1}{3 \cdot 4}$

$a_3 = 1 + \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$

$a_3 = 1 + \left(\frac{1}{1} - \frac{1}{4}\right)$

$a_n = 1 + \left(\frac{1}{1} - \frac{1}{n+1}\right)$

$= 1 + \frac{n}{n+1}$

$a_n = \frac{2n+1}{n+1}$

Q The solution of $a_n = a_{n-1} + (2n+1)$ where $a_0 = 1$ is

a) n^2 b) $(n+1)^2$

c) $(2n+1)^2$ d) $(2n-1)^2$

$n = 1$

$a_1 = a_0 + (2 \cdot 1 + 1)$

$a_1 = a_0 + 3$

$a_1 = 1 + 3$

$a_1 = 4$

Option b is giving 4

∴ B → correct option

~~$a_1 = a$~~ Long Method

$a_1 = a_0 + (2(1) + 1)$

$= 1 + 2(1) + 1 = (1+1)^2$

$a_2 = a_1 + 2(2) + 1$

$= 2^2 + 2(2) + 1 = (2+1)^2$

$a_n = (n+1)^2$

Q The solution of $a_n = n \cdot a_{n-1}$, where $a_0 = 1$ is

a) n^2 b) 2^n c) $n!$ d) $\frac{n(n+1)}{2}$

$a_n = n \cdot a_{n-1}$

∴ it is $n!$

Q.) The solution of the recurrence relation
$a_n = 4a_{n-1} + 3n 2^n$ where $a_0 = 4$ is

a) $a_n = 10(4^n) - (3n+6)2^n$

b) $a_n = 7(4^n) + (3n+4)2^n$

c) $a_n = 8(4^n) + (2n+3)2^n$

d) $a_n = 5(4^n) - (2n+6)2^n$

$n = 1$, $a_1 = 4a_0 + 3(1)2^1$
$= 4(4) + 6$
$= 22$

Option A giving $a_1 = 22$

∴ option A correct

Q The solution of
$\sqrt{a_n} - 2\sqrt{a_{n-1}} - \sqrt{a_{n-2}} = 0$
where $a_0 = 1$ & $a_1 = 2$

a) $a_n = \left[\frac{2^{n+1} + (-1)^n}{3}\right]^2$

b) $(n+1)^2$

c) $(n-1)^3$

d) $(n-1)^2$

* By using $a_0 = 1$

We can eliminate option C

as for $n = 0$, $a_0 = 1$

But here for $n = 0$, $a_0 = -1$

* By using $a_1 = 2$

We can eliminate option B & C

b option → $(1+1)^2 = 4$
d " → $(1-1)^2 = 0$

a option → ✓

{ (1,2)(2,3)(3,4)(5,4)(1,3)(2,4)(1,4)

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Date	~~Date~~ Topic	Expected Marks	Target

Step 1 : Send Scan Copy of problem

Step 2 : Write your analysis

Step 3 : Is my analysis correct

AxA = { ~~[illegible]~~ {1}, {2}, {1,2}, ~~[illegible]~~ {2,1}, } ~~[illegible]~~

(1,1)(2,2)(3,3) (1,3)(3,1)

(1,1)(2,2)(3,3) (1,2)(2,1)

[2] = {2, 5}

[3] = {3, 1}

[4] = {4, }

[5] = {5, 2}

[1] = {1, 5, , }

[2] = {2, 3, 6, }

(0,1)
(1,2)
(2,3)

2^n

2^{n-2}

${}^nC_2$

Oh My Gold...

[3] = {2, 3, 6}

[4] = {4, }

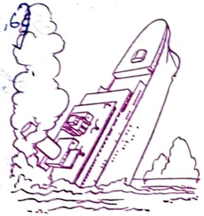
One ounce of gold can be stretched to a length of 80 kilometres and the resulting wire would be just 5 micron wide.

Gold weighs 19.3 times more then an equal volume of water

---

[5] = {1, 5} [6] = {2, 3, 6}

Including the four smokestacks, the Titanic was as will as a 17 -storey building !



Titanic

The ship's engines used more then 800 tonnes of coal each day!

Source : Internet

20:00

Methods of Characteristic Roots

Consider Linear Recurrence Relation

$\Rightarrow l_0 a_n + l_1 a_{n-1} + l_2 a_{n-2} + \dots + l_k a_{n-k} = f(n)$ — ①

Replace: $(n = n+k)$

$\Rightarrow l_0 a_{n+k} + l_1 a_{n+k-1} + l_2 a_{n+k-2} + \dots + l_k a_n$
$= f(n+k)$

$\Rightarrow l_0 E^k(a_n) + l_1 E^{k-1}(a_n) + l_2 E^{k-2}(a_n) + \dots + l_k E^0(a_n)$
$= f(n)$

Shift Operator

$\because E^k(a_n) = a_{n+k}$

$E^2(a_n) = a_{n+2}$

$\Rightarrow (l_0 E^k + l_1 E^{k-1} + l_2 E^{k-2} + \dots + l_k) a_n = f(n)$

$\phi(E) a_n = f(n)$ —— ⊛

Case i) :-
If $f(n) = 0$, then $\phi(E) a_n = 0$ is called Homogeneous Equation

$a_n = C \cdot F$

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Case ii) If $f(n) \neq 0$, then $\phi(E) a_n = f(n)$ is called non-homogeneous equation

$$a_n = C.F + P.S$$

[Complementar function ⟶ C.F
Particular solution ⟶ P.S]

$\phi(E) a_n = 0$

where $\phi(E) = l_0 E^k + l_1 E^{k-1} + l_2 E^{k-2} + \dots + l_k$

Rules for finding Complementary function (C.F)

1.) If Roots are real and distinct say $t_1, t_2, t_3, \dots t_k$

$$C.F = C_1 t_1^n + C_2 t_2^n + \dots + C_k t_k^n$$

2.) If Roots are real and two roots are equal say $t_1, t_1, t_3, t_4, \dots, t_k$.

$$C.F = (C_1 + C_2 n) t_1^n + C_3 t_3^n + C_4 t_4^n + \dots + C_k t_k^n$$

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3) If Roots are real and three roots are equal say $t_1, t_1, t_1, t_4, t_5, \dots, t_k$

$$C.f = (C_1 + C_2 n + C_3 n^2) t_1^n + C_4 t_4^n + C_5 t_5^n + \dots + C_k t_k^n$$

4.) If pair of roots are complex say $\alpha \pm i\beta$ then,

$$C.f = r^n \{C_1 \cos(n\theta) + C_2 \sin(n\theta)\}$$

where $r = \sqrt{\alpha^2 + \beta^2}$

$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

[Not in GATE syllabus]

Q.1 $a_n - 2a_{n-1} + a_{n-2} = 0$

Replace 'n' by n+2

$a_{n+2} - 2a_{n+1} + a_n = 0$ $[E^2 - 2E + 1 = 0]$ Shift Operator

$t^2 - 2t^1 + 1 = 0$ $[E = t]$

$(t-1)^2 = 0 \Rightarrow t_1 = t_2 = 1$

$C.f = (C_1 + C_2 n) t_1^n$

$= (C_1 + C_2 n) 1^n$

$\therefore a_n = C_1 + C_2 n$

Q $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$

Replace 'n' by n+3

$$a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0$$

$$t^3 - 3t^2 + 3t^1 - 1 = 0$$

$$(t-1)^3 = 0$$

$$\Rightarrow t_1 = t_2 = t_3 = 1$$

$$a_n = (c_1 + c_2 n + c_3 n^2) 1^n$$

Q $a_n - 5a_{n-1} + 6a_{n-2} = 0$

Replace 'n' by n+2

$$a_{n+2} - 5a_{n+1} + 6a_n = 0$$

$$E^2(a_n) - 5E^1(a_n) + 6a_n = 0$$

$$(E^2 - 5E + 6) a_n = 0$$

$$\boxed{\phi(E) a_n = F(n)}$$

here $\phi(E) = E^2 - 5E + 6$

Ch Characteristic Eqn. is

$$\boxed{\phi(t) = 0}$$

$$t^2 - 5t + 6 = 0$$

$$(t-3)(t-2) = 0$$

$$t_1 = 3, \quad t_2 = 2$$

$$C.F = c_1 t_1^n + c_2 t_2^n$$

$$\boxed{a_n = C.F = c_1 3^n + c_2 2^n}$$

* $(t-2)^2 (t-3) = 0$

$$t_1 = t_2 = 2, \quad t_3 = 3$$

$$\therefore a_n = C.F = (c_1 + c_2 n) 2^n + c_3 3^n$$

* $(t-3)^3 (t-2) = 0$

$$a_n = C.F = (c_1 + c_2 n + c_3 n^2) 3^n + c_4 2^n$$

Rules for finding Particular Solution (P.S) :- $\phi(E) a_n = f(n)$

when $f(n) = b^n$

then $P.S = \frac{1}{\phi(b)} \cdot b^n$,

where $\phi(b) \neq 0$

Case of Failure (If $\phi(b) = 0$) :

If $\phi(E) = (E-b)^k$ then

$P.S = {}^nC_k \, b^{n-k}$

Q $a_n - 3a_{n-1} = 2^n$

'n' by $n+1$

$a_{n+1} - 3a_n = 2^{n+1}$

~~(E~~ $\phi(E) a_n = f(n)$

$(E-3)a_n = 2^{n+1}$

$C.f = C_1 3^n$

Again

$(E-3)a_n = 2^{n+1}$

where $\phi(E) = E - 3$

as the format is b^n

$f(n) = 2^{n+1} = 2 \cdot 2^n$

here $b = 2$
$\phi(b) = \phi(2)$
$= 2 - 3 \neq 0$

$P.S = \frac{1}{\phi(b)} \times b^n$

$= 2\left[\frac{1}{-1} 2^n\right]$

$= -2^{n+1}$

$G.S = C.F + P.S$ [G.S → General solution]

$a_n = C_1 \cdot 3^n - 2^{n+1}$

Q $a_n - 5a_{n-1} + 6a_{n-2} = 2^n$

Replace 'n' by $n+2$

$a_{n+2} - 5a_{n+1} + 6a_n = 2^{n+2}$

$(E^2 - 5E + 6) a_n = 2^{n+2}$

$C.F = C_1 2^n + C_2 3^n$

here $\phi(E) = (E-3)(E-2)$

$f(n) = 2^{n+2} = b^n$

$P.S = \frac{1}{\phi(E)} \cdot b^n$

$= \frac{1}{(E-3)(E-2)} \times 2^{n+2}$

$= 2^2\left[\frac{1}{(E-3)(E-2)} 2^n\right]$

$= 2^2\left[\frac{1}{(E-3)}\left(\frac{1}{(E-2)} 2^n\right)\right]$

If $\phi(b) = 0$

i.e $\phi(E) = (E-b)^k$

then $P.S = {}^nC_k b^{n-k}$

here $k = 1, b = 2$

$= 2^2\left[\frac{1}{(E-3)}\left({}^nC_1 2^{n-1}\right)\right]$

$= 2^2\left[\frac{n}{2}\left(\frac{1}{E-3} \times 2^n\right)\right]$

If $\phi(b) \neq 0$

$P.S = \frac{b^n}{\phi(b)}$

here $\phi(E) = (E-3)$, $b = 2$

$\therefore \phi(2) = 2 - 3 \neq 0$

$= 2^2\left[\frac{n}{2}\left[\frac{2^n}{-1}\right]\right]$

$= -2^{n+1} \cdot n$

$G.S = C.F + P.S$

$a_n = C_1 2^n + C_2 3^n - 2^{n+1} \cdot n$

Q The solution of $a_n - 3a_{n-1} + 2a_{n-2} = 2^n$ is ______

a) $C_1 + C_2 2^n + 2n(2^n)$

b) $C_1 - C_2 n - 2n(2^n)$

c) $C_1 + C_2 n + n(2^n)$

d) $C_1 + C_2 n + 3n(2^n)$

$a_{n+2} - 3a_{n+1} + 2a_n = 2^{n+2}$

$\phi(E) = E^2 - 3E + 2$

$f(n) = 2^2 \cdot 2^n = b^n$ [∵ $b=2$]

∴ $C.F = C_1 \cdot 2^n + C_2 1^n$

$P.S = \frac{1}{\phi(E)} \times f(n)$

$= \frac{1}{(E-2)(E-1)} 2 \cdot 2^n$

$= 2^2 \left[\frac{1}{(E-2)}\left(\frac{1}{(E-1)} 2^n\right)\right]$

[here $b=2$]

$= 2^2 \left[\frac{1}{(E-2)^1} \times \frac{2^n}{1}\right]$

(If $\phi(b) \neq 0$ then, $P.S = \frac{b^n}{\phi(b)}$)

$= 2^2 \left[{}^nC_1 \cdot 2^{n-1}\right]$

(∵ If $\phi(b) = 0$
If $\phi(E) = (E-b)^k$
∴ ${}^nC_k \, b^{n-k}$)

$= 2^2 n \cdot 2^{n-1}$

$P.S = 2n \cdot 2^n$

$G.S = C.F + P.S$

$= C_1 \cdot 2^n + C_2 + 2n \cdot 2^n$

option - A

Q The solution of $a_n - 6a_{n-1} + 9a_{n-2} = 3^n$ is ______

a) $(C_1 + C_2 n)3^n + [n(n-1)/2]3^n$

b) $(C_1 + C_2 n)3^n + n(n+1)3^n$

c) $(C_1 + C_2 n)3^n + n(n-1)3^{n-1}$

d) $(C_1 + C_2 n)3^n + n(n-1)3^{n-2}$

$\because C.F = (C_1 + C_2 n) 3^n$

$(E^2 - 6E + 9) a_n = 3^{n+2}$

$P.S = \frac{1}{\phi(E)} \times f(n)$

$\because \phi(E) = E^2 - 6E + 9$

$= (E-3)^2$

$f(n) = 3^n \cdot 3^2$

$\overset{\uparrow}{\text{here } b} = 3$

$= \frac{1}{(E-3)^2} \times 3^2 \cdot 3^n$

$= 3^2 \left[\frac{1}{(E-3)^2} \times 3^n \right]$

$\phi(b) = \phi(3) = (3-3)^2 = 0$

$\therefore {}^nC_k \, b^{n-k}$

$= 3^2 \left[{}^nC_2 \, 3^{n-2} \right]$

$= {}^nC_2 \, 3^n$

$= \frac{n(n-1)}{2} \times 3^n$

$\therefore a_n = (C_1 + C_2 n) 3^n + \frac{n(n-1)}{2} 3^n$

option A → correct

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Lecture 13A

Method of finding Undetermined Coefficients

→ K → Degree of polynomial

1) when $f(n) = b^n n^k$ $(k = 1, 2, 3, \ldots)$ and 'b' is not characterstic root then we can choose

$$P.S = b^n (A_0 n^k + A_1 n^{k-1} + A_2 n^{k-2} + \ldots + A_k)$$

2) when $f(n) = b^n \cdot n^k$ $(k = 1, 2, 3, \ldots)$ and 'b' is a characterstic root with multiplicity 'm' then we choose

$$P.S = b^n (A_0 n^k + A_1 n^{k-1} + A_2 n^{k-2} + \ldots + A_k) n^m$$

Where $A_0, A_1, A_2, \ldots, A_k$ are called undetermined coefficients.

Q $a_n - a_{n-1} = n$

Replace 'n' by n+1

$a_{n+1} - a_n = n+1$

$(E-1) a_n = n+1$

Characterstic root $t_1 = 1$

$C.F = C_1 t_1^n = C_1 (1)^n$

→ $\phi(E) a_n = f(n)$

here $\phi(E) = E - 1$

$f(n) = n + 1 = 1^n (n^1 + 1) = b^n . n^k$.

here $b = 1$, $K = 1$

here $b = 1$, is char. root with multiplicity $m = 1$.

∴ $P.S = b^n (A_0 n^k + A_1 n^{k-1} + \dots + A_n) . n^m$

$P.S = 1^n (A_0 n^1 + A_1) n^1$

$P.S = A_0 n^2 + A_1 n$

Q $(E-1)^2 a_n = n^2 + 1$

Char. root, $t_1 = t_2 = 1$

$C.F = (C_1 + C_2 n) 1^n$

here, $f(n) = n^2 + 1 \Rightarrow 1^n (n^2 + 1) = b^n . n^k$

$b = 1$, $K = 2$.

$b = 1$ is a char root with multiplicity $m = 2$.

$P.S = b^n (A_0 n^k + A_1 n^{k-1} + \dots + A_n) n^m$

$= 1^n (A_0 n^2 + A_1 n + A_2) n^2$

$= A_0 n^4 + A_1 n^3 + A_2 n^2$

Q $a_n - 2a_{n-1} + a_{n-2} = 3^n n$

Replace 'n' by $n+2$

$a_{n+2} - 2a_{n+1} + a_n = 3^{n+2} (n+2)$

$(E^2 - 2E + 1) a_n = 3^{n+2} (n+2)$

$(E-1)^2 a_n = 3^{n+2} (n+2)$ ---- ①

Char. roots, $t_1 = t_2 = 1$

→ $\phi(E) a_n = f(n)$

here $\phi(E) = E^2 - 2E + 1$

$f(n) = 3^{n+2} (n^1 + 2) = b^n . n^K$

∴ $b = 3$, $K = 1$

here 'b' is not a char. root

~~P.S ≠ $b^n (A$~~

$P.S = b^n (A_0 n^k + A_1 n^{k-1} + \dots + A_k)$

$= 3^n (A_0 n + A_1)$

GATE-04

Q The recurrence equation

$T(1) = 1$

and $T(n) = 2T(n-1) + n$ $(n \geq 2)$

evaluates to

a) $2^{n+1} - n - 2$ b) $2^n - n$

c) $2^{n+1} - 2n - 2$ d) $2^n + n$

Let $T(n) = a_n$, $T(1) = a_1 = 1$

$\therefore a_n - 2a_{n-1} = n \longleftarrow ⊛$

$a_{n+1} - 2a_n = n+1$

$(E-2) a_n = n+1$

Char root $= 2$

$\therefore$ C.F $= c_1 2^n$

$\phi(E) a_n = f(n)$

$\phi(E) = E - 2$

$f(n) = 1^n (n^1 + 1) = b^n . n^k$

here $b = 1$, $k = 1$

here 'b' is not char. root

$\therefore P.S = b^n (A_0 n^k + A_1 n^{k-1} + \dots + A_k)$

$a_n = 1^n (A_0 n + A_1)$

$a_n = A_0 n + A_1$

Substitute P.S(a_n) in eq^n ⊛

$(A_0 n + A_1) - 2(A_0(n-1) + A_1) = n$

$(\because ⊛ \Rightarrow a_n - 2a_{n-1} = n)$

$A_0 - 2A_0 = 1 \rightarrow A_0 = -1$

$A_1 + 2A_0 - 2A_1 = 0$

$A_1 = 2A_0$

$A_1 = 2(-1) = -2$

$P.S = A_0 n + A_1 = -1(n) + (-2) = -n - 2$

$\therefore$ G.S $a_n = C.F + P.S$

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~~P.S = b^n(A~~

$P.S = b^n (A_0 n^k + A_1 n^{k-1} + \ldots + A_k)$

$= 3^n (A_0 n + A_1)$

GATE-04

Q The recurrence equation

$T(1) = 1$

and $T(n) = 2T(n-1) + n$ $(n \geq 2)$

evaluates to

a) $2^{n+1} - n - 2$ b) $2^n - n$

c) $2^{n+1} - 2n - 2$ d) $2^n + n$

Let $T(n) = a_n$, $T(1) = a_1 = 1$

$\therefore a_n - 2a_{n-1} = n$ ⟶ ⊛

$a_{n+1} - 2a_n = n+1$

$(E-2) a_n = n+1$

Char root $= 2$

$\therefore C.F = C_1 . 2^n$

$\phi(E)\, a_n = f(n)$

$\phi(E) = E - 2$

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$f(n) = 1^n (n^1 + 1) = b^n . n^k$

here $b = 1$, $k = 1$

here 'b' is not char. root

$\therefore P.S = b^n (A_0 n^k + A_1 n^{k-1} + \ldots + A_k)$

$a_n = 1^n (A_0 n + A_1)$

$a_n = A_0 n + A_1$

Substitute P.S(a_n) in eqn ⊛

$(A_0 n + A_1) - 2(A_0 (n-1) + A_1) = n$

$(\because ⊛ \Rightarrow a_n - 2a_{n-1} = n)$

$A_0 - 2A_0 = 1 \rightarrow A_0 = -1$

$A_1 + 2A_0 - 2A_1 = 0$

$A_1 = 2A_0$

$A_1 = 2(-1) = -2$

$P.S = A_0 n + A_1 = -1(n) + (-2) = -n - 2$

$\therefore$ G.S $a_n = C.F + P.S$

~~$P.S = b^n (A$~~

$P.S = b^n (A_0 n^K + A_1 n^{K-1} + \dots + A_K)$

$= 3^n (A_0 n + A_1)$

GATE-04

Q The recurrence equation

$T(1) = 1$

and $T(n) = 2T(n-1) + n$ $(n \geq 2)$

evaluates to

a) $2^{n+1} - n - 2$ b) $2^n - n$

c) $2^{n+1} - 2n - 2$ d) $2^n + n$

let $T(n) = a_n$, $T(1) = a_1 = 1$

$\therefore a_n - 2a_{n-1} = n$ —— ⊛

$a_{n+1} - 2a_n = n + 1$

$(E - 2) a_n = n + 1$

Char root $= 2$

$\therefore C.F = C_1 \cdot 2^n$

$\phi(E) a_n = f(n)$

$\phi(E) = E - 2$

$f(n) = 1^n (n^1 + 1) = b^n \cdot n^K$

here $b = 1$, $K = 1$

here 'b' is not char. root

$\therefore P.S = b^n (A_0 n^K + A_1 n^{K-1} + \dots + A_K)$

$a_n = 1^n (A_0 n + A_1)$

$a_n = A_0 n + A_1$

Substitute $P.S(a_n)$ in eqn ⊛

$(A_0 n + A_1) - 2(A_0(n-1) + A_1) = n$

$(\because ⊛ \Rightarrow a_n - 2a_{n-1} = n)$

$A_0 - 2A_0 = 1 \rightarrow A_0 = -1$

$A_1 + 2A_0 - 2A_1 = 0$

$A_1 = 2A_0$

$A_1 = 2(-1) = -2$

$P.S = A_0 n + A_1 = -1(n) + (-2) = -n - 2$

$\therefore$ G.S $a_n = C.F + P.S$

$a_n = C_1 \cdot 2^n - n - 2$

Put n=1.

$a_1 = C_1 \cdot 2 - 1 - 2$ (we know $a_1 = 1$)

$1 = 2C_1 - 3$

$C_1 = 2$

$\therefore a_n = 2(2^n) - n - 2$

$$\boxed{a_n = 2^{n+1} - n - 2}$$

Q The solution of ~~$a_n - 2a_n$~~ $a_n - 2a_{n-1} + a_{n-2} = 3n+5$ is ______.

H.W

a) $C_1 + C_2 n + 4n^2 + \frac{n^3}{2}$

b) $C_1 + C_2 n + 2n^2 + \frac{n^3}{2}$

c) $C_1 + C_2 n + n^2 + \frac{n^3}{2}$

d) $C_1 + C_2 n + 3n^2 + \frac{n^3}{2}$

Q The solution of $a_n - 2a_{n-1} = 3^n(n+2)$ —①

$C.F = C_1 \cdot 2^n$ here characteristic root is '2'

$f(n) = 3^{n+1}(n'+3) = b^n \cdot n^k$,

here $b = 3, k = 1$

'b' is not char. root

so $P.S = b^n(A_0 n' + A_1)$

$a_n = 3^n(A_0 n + A_1)$

Substitute P.S in eqn —①

$3^n(A_0 n + A_1) - 2 \cdot 3^{n-1}(A_0(n-1) + A_1) = 3^n(n+2)$

$(A_0 n + A_1) - \frac{2}{3}(A_0(n-1) + A_1) = (n+2)$

(Cancel: 3^n)

Put n = 1

$A_0 + A_1 - \frac{2}{3} A_1 = 3$

$\Rightarrow A_0 + \frac{A_1}{3} = 3$ —②

Put n = 0

$A_1 + \frac{2}{3}A_0 - \frac{2}{3}A_1 = 2$

$\Rightarrow \frac{2A_0}{3} + \frac{A_1}{3} = 2 \quad -③$

By solving eqⁿ ② & ③

$\frac{A_0}{3} = 1$

$A_0 = 3$

$A_1 = 0$

$\therefore P.S = 3^n(A_0 n + A_1)$

$= 3^n(3n)$

$\therefore a_n = C.F + P.S$

$= C_1 \cdot 2^n + 3^n(3n)$

FLOOR & CEIL CONCEPT

$\lceil 2.1 \rceil = 3$ $\lfloor 2.1 \rfloor = 2$

$\lceil 2.5 \rceil = 3$ $\lfloor 2.5 \rfloor = 2$

$\lceil 2.9 \rceil = 3$ $\lfloor 2.9 \rfloor = 2$

~~Pieg~~ Pigeon Hole PRINCIPLE

Q.) what is the minimum number of persons we have to choose randomly so that atleast 9 person were born in same month?

Jan Feb March April May June July

J	F	M	A	M	Ju	Jl	Au	Sept	Oct	N	Dec
8	8	8	8	8	8	8	8	8	8	8	8

If we distribute 96 people such that every 8 person have B'day in same month then ~~the~~ if we choose one more person we can definitely say atleast 9 person were born in same month.

$\therefore$ ~~[illegible]~~ $8(12) + 1 = 97$

Minimum No. of people = 97.

Q A bag contains 6 Red balls, 8 blue balls, 10 green balls, 15 white balls, 20 yellow balls. what is the minimum number of balls we have to choose randomly from the bag to ensure that we get atleast 6 balls of same color?

Red	6	5
Blue	8	5
Green	10	5
White	15	5
Yellow	20	5

25+1=26

Pigeon hole ~~formula~~ Concept

$k+1 = 6$, $\Rightarrow k=5$

$n=5$ (no. of Colours / Pigeon holes)

Minimum $= kn+1$

$= 5(5)+1$

$= 26$.

Q In previous question what is the minimum no. of balls we have to choose randomly from the bag to ensure that we get atleast ~~6~~ 9 balls of same color?

Red	6	6
Blue	8	8
Green	10	8
White	15	8
Yellow	20	8

38+1=37

$k+1 = 9$

$k=8$, $n=3$

$= (6+8) + (3 \times 8 + 1)$

$= 39$

→ By Pigeon hole Principle

Pigeonhole Principle

If there are 'n' pigeon holes, if we distribute kn+1 pigeons then Average no. of pigeons per pigeonhole

$$A = \left[\frac{kn+1}{n}\right] = k + \frac{1}{n}$$

Rule :-

If we distribute (kn+1) piegons among 'n' pigeon holes then

i) Some piegon holes contain atleast

$$\lceil A \rceil = \left\lceil k + \frac{1}{n} \right\rceil = k+1$$

ii) Some ~~piegon~~ pigeon holes contains atmost

$$\lfloor A \rfloor = \left\lfloor k + \frac{1}{n} \right\rfloor = k \text{ pigeons}$$

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Note: Minimum no. of pigeons are required to distribute to ensure that some pigeons holes contains atleast $(k+1)$ pigeons $= Kn+1$

~~2:0:0~~ ~~2=0~~ ~~Kn+1~~

2:06:58

Q A box contain 12 Red, 7 Blue and x Green balls. If the minimum number of balls we have to choose randomly from the box to gurantee that we have 6 balls of same colour is 15, then $x =$ ______.

Red $= 12 \rightarrow 5$
Blue $= 7 \rightarrow 5$
Green $= x \rightarrow$

$Kn+1 = 15$, $x = ?$

Case (i) If $x \geq 15$

Red $= 5$
Blue $= 5$
Green $= 5$
$= 15 + 1$
$= 16$

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$\therefore$ If $x \geq 15$, then minimum is 16 which is contradiction to minimum is 15.

Case ii)
Red $= 12$ —— 5
Blue $= 7$ —— 5
Green $= x(<5)$ —— x

$$(5+5+x)+1 = 15$$
$$x = 4$$

Q Suppose thirty buses are used to transport 2000 students and each bus has 80 seats. Consider the following statements.

S_1: One of the buses will carry atleast 67 passengers.

S_2: One of the buses will have atleast 14 empty seats.

Which of the following ~~stats~~ is true?

a) S_1 is TRUE and S_2 is FALSE.
b) S_1 is FALSE and S_2 is TRUE.
c) both S_1 and S_2 are TRUE.
d) both S_1 and S_2 are false.

2:23

$$A = \frac{2000}{30} = 66.6$$

Some buses will atleast $\lceil A \rceil = \lceil 66.6 \rceil$
$= 67$

Atmost $\lfloor A \rfloor = \lfloor 66.6 \rfloor = 66$
passengers

Option C → Correct

Q A palindrome is a string whose reversal is identical to the string. How many bit strings of length 'n' are palindromes?

a) 2^{n-1} b) $2^{n/2}$
c) $2^{\lceil n/2 \rceil}$ d) $2^{\lfloor n/2 \rfloor}$

If 'n' is even length

$$\overbrace{\underbrace{\underline{1}\ \underline{1}\ \underline{0}}_{n/2}\ \Big|\ \underline{0}\ \underline{1}\ \underline{1}}^{n}$$

$$\underbrace{\underline{1/0}\ \ \underline{1/0}\ \ \underline{1/0}}\ \Big|\ \_\ \ \_\ \ \_$$

$$2^{n/2} \times 1$$

If 'n' is odd length

$$\underbrace{\_\ \ \_\ \ \_}_{\frac{n-1}{2}}\ \ \boxed{1/0}\ \ \_\ \ \_\ \ \_$$

$$2^{\frac{n-1}{2}} \times 2 \times 1$$

$$= 2^{\frac{n+1}{2}}$$

$n \to$ even: $2^{n/2}$
$n \to$ odd: $2^{n+1/2}$
$\to 2^{\lceil \frac{n}{2} \rceil}$

Q In a binary matrix each element is 0 or 1. How many symmetric binary matrices of order 3x3 are possible?

$$\{\underbrace{\overset{1/0}{(1,1)}\overset{1/0}{(2,2)},\dots,\overset{1/0}{(n,n)}}_{n}\overbrace{\underbrace{\overset{0/1}{(1,2)(2,1)}\overset{0/1}{(1,3)(3,1)},\dots\overset{0/1}{(1,n)(n,1)}}}^{\frac{n(n-1)}{2}}\}$$

$$2^n \times 2^{\frac{n(n-1)}{2}}$$

here $n = 3$

$\therefore = 2^3 \times 2^{3(2)/2} = 2^6 = 64$

Q A set S has n elements. How many ways we can choose subsets P and Q of S so that $(P \cap Q) = \phi$?

a) 3^{n-1} b) $3^n - 1$
c) 3^n d) $3^n + 1$

Q Suppose we have 6 different English movies, 8 different Telgu movies and 10 different Hindi movies. How many ways we can choose 2 movies of different language.

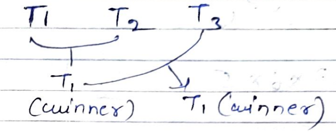
Sol EH (or) HT (or) ET

$$= ({}^{6}C_1 \times {}^{10}C_1) + ({}^{10}C_1 \times {}^{8}C_1) + ({}^{6}C_1 \times {}^{8}C_1)$$

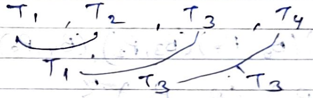
$$= 60 + 80 + 48 = 188$$

Q Suppose n players were enrolled in a singles elimination tennis tournament. How many matches are to be conducted to decide the winner?

a) $n-1$ b) $n+1$
c) 2^n d) 2^{n-1}



For 3 teams, 2 matches are conducted

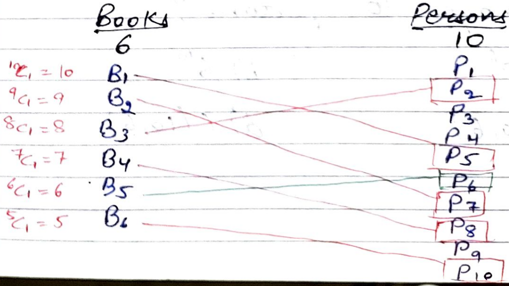


For 4 teams, 3 matches are conducted

∴ for n teams, (n−1) matches are to be conducted.

Option A → correct

Q How many ways 6 different books can be distributed among 10 persons so that no persons can take more than one book and maximum number of books are to be distributed?



$= {}^{10}C_1 \times {}^{9}C_1 - - - \times {}^{5}C_1$

$= 10 \times 9 \times 8 \times 7 \times 6 \times 5$ Ans

Method 2

$f : B \to P;$

$f : \{(B_1, P_5)(B_2, P_7)(B_3, P_2), (B_4, P_8)$
$(B_5, P_6)\ (B_6, P_{10})\}$

$|A| = m \quad , \quad |B| = n$

No. of one-one $= {}^{|B|}P_{|A|} = {}^{n}P_{m}$

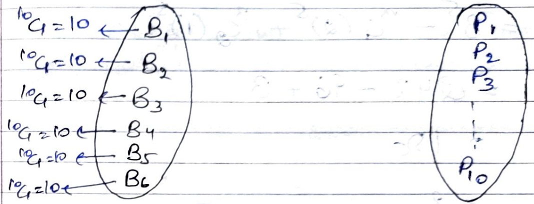
$${}^{10}P_6 = \frac{10!}{(10-6)!} = 10 \times 9 \times 8 \times 7 \times 6 \times 5$$

Q In how many ways can we distribute 6 different books among 10 persons?

a) 10^6 b) 6^{10}

c) P(10, 6) d) C(10, 6)

No restriction that each person can get only one book



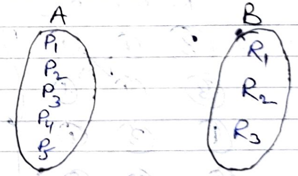
$= 10 \times 10 \times 10 \times 10 \times 10 \times 10$

$= 10^6$

$|A| = m = 6 \quad , \quad |B| = n = 10$

No. of function $= |B|^{|A|} = 10^6$

Q Number of ways we can assign 5 persons into 3 different rooms so that each room contains atleast one person is ______



$|A| = m = 5 \quad , \quad |B| = n = 3$

No. of onto $f^n = n^m - {}^{n}C_1 (n-1)^m + {}^{n}C_2 (n-2)^m$
$- - - - + (-1)^{n-1}\ {}^{n}C_{n-1} (1)^m$

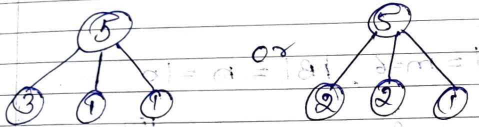
$$= 3^5 - {}^3C_1 (2)^5 + {}^3C_2 (1)^5$$

$$= 243 - 96 + 3$$

$$= 150$$

Method 2

Divide & Distribution



$$\left(\frac{5!}{3!\,1!\,1!\,2!} + \frac{5!}{2!\,2!\,2!} \right) \times 3!$$

$$= 150$$

R_1	R_2	R_3
3	1	1
3	1	1
1	3	1
1	3	1
1	1	3
1	1	3

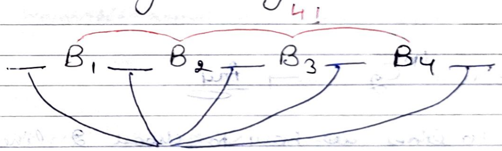
3!

Q How many 5-digit positive integers are possible with the digit 2, 3 or 5?

$$\underbrace{\underline{2/3/5}\quad \underline{2/3/5}\quad \underline{2/3/5}\quad \underline{2/3/5}\quad \underline{2/3/5}}$$

$$3^5 = 243$$

Q How many ways 4 boys and 4 girls can sit in a row so that no two girls are sitting side by side.



$${}^5C_4 \times 4!$$

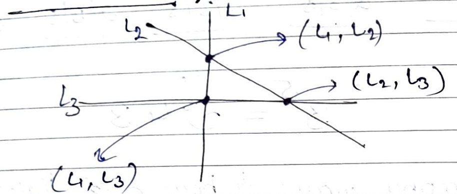
$$= 4! \times {}^5C_4 \times 4!$$

$$= 24 \times \frac{5!}{4!\,1!} \times 4!$$

$$= 24 \times 120$$

$$= 2880$$

Q The maximum number of points of intersection possible with 10 straight lines in a plane is



$^3C_2 = 3$

∴ $^{10}C_2$ → Ans

In 10 lines we have to choose 2 lines to make a intersection.
∴ $^{10}C_2$

$^{10}C_2 = 45$

GRAPH THEORY

1.) Basics
Graph, Null Graph, Trivial graph, parallel edges, multigraph, Connected graph, Complete graph, cycle graph, wheel graph, Bipartite graph, Complete Bipartite graph, Star graph.

★★★★★

2.) Sum degrees of vertices Theorem
Properties, Havel Hakimi Result, Planar graph

3.) Graph Coloring:
- Chromatic Number, Four-colour Theorem, Welc-Powel's algorithm

4.) Matching
- Basics, Matching Number, Perfect Matching

5.) Covering
- Vertex Covering Number, Line covering Number.

6.) Independent Set:
- Independent Set Number

★★★★★

7.) Connectivity :-
- Cut vertix, cut edge (Bridge)
- Edge connectivity, Vertix Connectivity
- Properties of Connectivity.

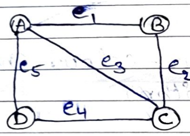
8.) Hamiltonian Circuit, Hamiltonian Path, Euler circuit and Euler path.

9) Isomorphic Graph

10) Basics of Spanning Tree

28/4/21 Lecture 14-A

Graph :-

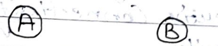


$E = \{e_1, e_2, e_3, e_4, e_5\}$

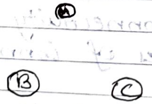
$V = \{A, B, C, D\}$

$G = (\underset{\text{Vertices}}{V}, \underset{\text{Edges}}{E})$

- Null Graph:-
  A graph with no edges.



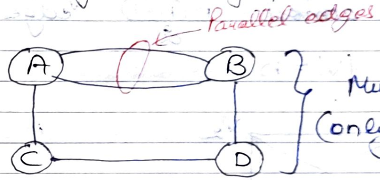
Null graph of order '2'.



Null graph of order '3'

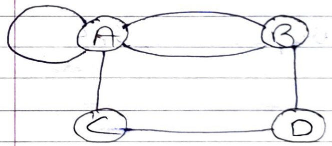
- Trivial Graph
  A null graph of order '1' is called Trivial Graph.

- Parallel edges, Multigraph



Multigraph (only parallel edges)

- Pseudo Graph :



(Parallel edges + Self loops)

Simple Graph

- No self loops
- No parallel edges

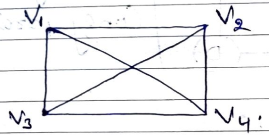
Degree → No. of edges coming from that vertex.

* Properties:-

1) Every simple connected graph must have atleast (n-1) edges where, $|V| = n$

2.) Every simple connected graph must have atmost $\frac{n(n-1)}{2}$ edges.

Where $|V| = n$.



$\{v_1, v_2, v_3, v_4, ------, v_n\}$

↓

$deg(v_1) = n-1$

$deg(v_2) = n-1$

⋮

$deg(v_n) = n-1$

$$\Rightarrow \sum_{i=1}^{n} deg(v_i) = (n-1) + (n-1) + \ldots + (n-1)$$

$$\Rightarrow 2|E| = n(n-1)$$

$$|E| = \frac{n(n-1)}{2}$$

22:56

3.) In a simple graph, number of odd degree vertices are always even

$\{v_1, v_2, v_3, ------, v_n\}$

$deg(v_1) + deg(v_2) + deg(v_3) + \ldots + deg(v_n)$

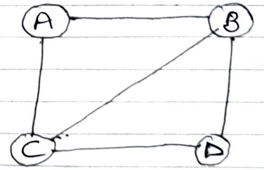
$= 2|E| = \text{even}$

(Hand Shaking Lemma)

Sum degree of Vertices Theorem:

Let $G(V, E)$ be a graph where $V = \{v_1, v_2, v_3, ----, v_n\}$

$$\sum_{i=1}^{n} deg(v_i) = 2|E|$$



$deg(A) = 2$

$deg(B) = 3$ (odd)

$deg(C) = 3$ (odd)

$dig(D) = 2$

$= 10$

$= 2 \times |E|$

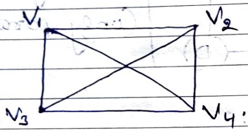
Degree → No. of edges coming from that vertex.

* Properties:-

1) Every simple connected graph must have atleast (n-1) edges where $|V| = n$

2.) Every simple connected graph must have atmost $\frac{n(n-1)}{2}$ edges.

where $|V| = n$.



$\{v_1, v_2, v_3, v_4, \ldots\ldots, v_n\}$

↓

$deg(v_1) = n-1$

$deg(v_2) = n-1$

⋮

$deg(v_n) = n-1$

$$\Rightarrow \sum_{i=1}^{n} deg(v_i) = (n-1) + (n-1) + \ldots + (n-1)$$

$$\Rightarrow 2|E| = n(n-1)$$

$$|E| = \frac{n(n-1)}{2}$$

22:56

3.) In a simple graph, number of odd degree vertices are always even

$\{v_1, v_2, v_3, \ldots\ldots, v_n\}$

$deg(v_1) + deg(v_2) + deg(v_3) + \ldots + deg(v_n)$

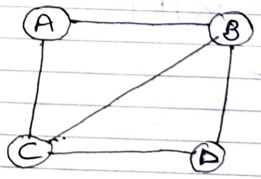
$= 2|E| = \text{even}$

(Hand Shaking Lemma)

Sum degree of Vertices Theorem:

Let $G(V,E)$ be a graph where $V = \{v_1, v_2, v_3, \ldots, v_n\}$

$$\sum_{i=1}^{n} deg(v_i) = 2|E|$$

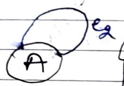


deg(A) = 2
deg(B) = 3 (odd)
deg(C) = 3 (odd)
dig(D) = 2

= 10
$= 2 \times |E|$

deg(v1) = 1
deg(v2) = 1

∴ e1 contributing two degree one degree for v1, and one degree for v2



deg(A) = 2,
here, e2 is contributing two degrees for vertex A

∴ Every edge contributing two degree but total no. of edges in G(V,E) = |E|

∴ Sum of degrees of all vertices = 2|E|

$$\sum_{i=1}^{n} deg(v_i) = 2|E|$$

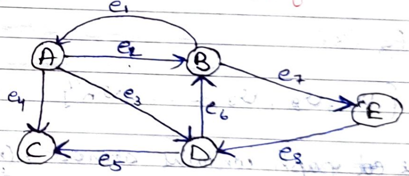
$$\sum_{i=1}^{n} deg(\underset{\substack{\downarrow \\ \text{Region}}}{R_i}) = 2|E|$$

Sum degrees of Vertices Theorem for Directed graph

$$\sum_{i=1}^{n} deg^{+}(v_i) = \sum_{i=1}^{n} deg^{-}(v_i) = |E|$$

i.e

$$\underbrace{\sum_{i=1}^{n} deg^{+}(v_i)}_{\text{(Sum of Indegree of all vertice)}} + \underbrace{\sum_{i=1}^{n} deg^{-}(v_i)}_{\text{(Sum of outdegree of all vertices)}} = 2|E|$$



Vertix	Indegree	Outdegree
A	$deg^{+}(A) = 1$	$deg^{-}(A) = 3$
B	$deg^{+}(B) = 2$	$deg^{-}(B) = 2$
C	$deg^{+}(C) = 2$	$deg^{-}(C) = 0$
D	$deg^{+}(D) = 2$	$deg^{-}(D) = 2$
E	$deg^{+}(E) = 1$	$deg^{-}(E) = 1$

$$\sum_{i=1}^{n} deg^{+}(v_i) = 8 \qquad \sum_{i=1}^{n} deg^{-}(v_i) = 8$$

Pg-32 (Graph Theory)

workbook, Q.9, IT-2004

Q Sol

$|E| = 27$

$$\sum_{i=1} deg(v_i) = 2|E|$$

$$6(2) + 3(4) + x(3) = 2(27)$$

$$3x = 54 - 24$$

$$x = 10$$

∴ Total no. of vertices = 6+3+10
= 19

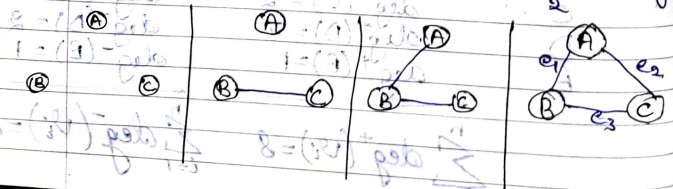
(D-option)

Q.3 Pg (31)

Sol $v = \{v_1, v_2, v_3, ----, v_n\}$

A simple ~~ed~~ graph contains atmost $\frac{n(n-1)}{2}$ edges (i.e e_1 e_2 ------ $e_{\frac{n(n-1)}{2}}$)

let n = 3, vertices, it has atmost $\frac{3(3-1)}{2} = 3$ edges



Every edges has 2 possibilities i.e (present or absent)

There are atmost $\frac{n(n-1)}{2}$ edges

∴ ~~Tota~~ $2^{\frac{n(n-1)}{2}}$ undirected graphs are possible.

Q.8 Pg (31)

Note: An acyclic undirected graph with maximum no. of edges is called a Tree but a Tree contains (n-1) edges where $|V| = n$

Option-A correct

Q 42 Pg - 35

No. of vertices of degree '1' = n_1

" " " degree '2' = n_2

" " degree '3' = n_3

degree 'K' = n_K

In a tree we have total no. of edges $= n-1$

here $|V| = n$,
but it is given that

$$|V| = n_1 + n_2 + \cdots + n_k$$

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

$\Rightarrow n_1(1) + n_2(2) + n_3(3) + \cdots + n_k(k) = 2(|V|-1)$

$\Rightarrow n_1 + 2n_2 + 3n_3 + \cdots + k \cdot n_k = 2(n_1 + n_2 + n_3 + \cdots + n_k - 1)$

$\Rightarrow n_1 + 2n_2 + 3n_3 + \cdots + kn_k = 2n_1 + 2n_2 + \cdots + 2n_k - 2$

$$\boxed{n_1 = n_3 + 2n_4 + \cdots + (k-2)n_k + 2}$$

Option C

Q.56 G.2017

$|V| = 10$

$\sum_{i=1}^{n} \deg(v_i) = 2|E|$ $\left(\because \text{In a Tree } |E| = |V|-1\right)$

$= 2(|V|-1)$

$= 2(10-1)$

$= 18$

GATE - 2010

Q20

$$\xi(G) = \sum_{d} i_d \times d$$

where i_d = No. of vertices of degree d

S and T are two different trees with

$$\xi(S) = \xi(T)$$

$$\sum_{d} i_d \times d = \sum_{d} i_d \times d$$

$|S| = |T|$ (Both tree contain same no. of edges)

no. of edges in Tree S.

1:31:20

. option C

Note :- Let $G(V,E)$ be a graph

$\delta(G)$ = Minimum degree of all vertices

$\Delta(G)$ = Maximum degree of all vertices

$A(G)$ = Average degree of all vertices

$= \dfrac{\text{Sum of all degree of all vertices}}{\text{Total no. of vertices}}$

$$= \frac{\sum_{i=1}^{n} deg(v_i)}{|V|}$$

$$= \frac{2|E|}{|V|}$$

Now $\delta(G) \le A(G) \le \Delta(G)$

$$\boxed{\delta(G) \le \frac{2|E|}{|V|} \le \Delta(G)}$$

Q.6

$|E| \le 3|V| - 6$ —①

We know that

$$\delta(G) \le \frac{2|E|}{|V|}$$

$$\le \frac{2(3|V|-6)}{|V|}$$

$$\delta(G) \le 6 - \frac{12}{|V|} \ne 6$$

Option D

2:14:30

Q.57 G.2017

$|E| = 25$, $|V| = n$

Each vertex having degree is atleast '3'

∴ $\delta(G) = 3$

$$\delta(G) \le \frac{2|E|}{|V|}$$

$$|V| \le \frac{2|E|}{\delta(G)}$$

$$|V| \le \frac{50}{3}$$

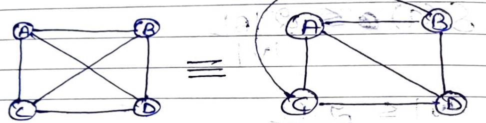
$$|V| \le 16.6$$

∴ Max $|V| = 16$

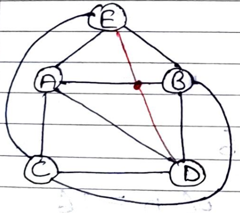
Max possible value of $n = 16$

Planar Graph

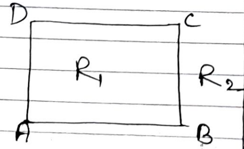
A multigraph which can be drawn on plane so that no two edges crossed with each other at non vertix point.



Planar graph



Non-planar graph



$deg(R_1)=4$

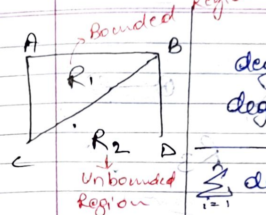
$deg(R_2)=4$

$\sum_{i=1}^{n} deg(R_i)=2|E|$

$4+4=2(4)$

$|V|+|R|=|E|+2$

$4+2=4+2$



$deg(R_1)=3$

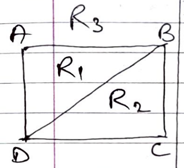
$deg(R_2)=5$

$\sum_{i=1} deg(R_i)=2|E|$

$3+(5)=2(4)$

$|V|+|R|=|E|+2$

$4+2=4+2$



$deg(R_1)=3$

$deg(R_2)=3$

$deg(R_3)=4$

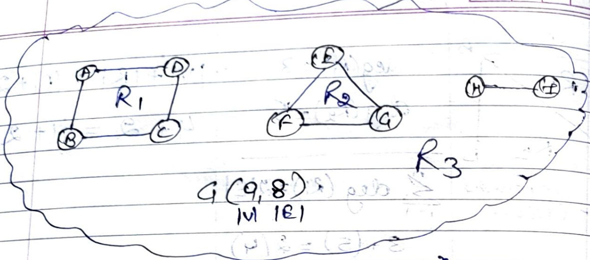
Euler's Formula

i) If $G(V,E)$ is a simple connected planar graph then

$$|V|+|R|=|E|+2$$

ii) If $G(V,E)$ is a simple planar graph with 'K' connected components

$$\therefore |V|+|R|=|E|+(K+1)$$



$|V| + |R| = |E| + (k+1)$

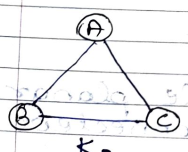
$9 + 3 = 8 + (3+1)$

* There are 3 components

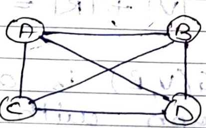
Complete Graph

i) Denoted K_n (If 'n' vertices are there)

ii) $|E(K_n)| = \frac{n(n-1)}{2}$

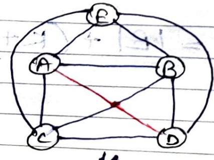


K_3



K_4

Non-planar Graph



K_5

It is non-planar graph with minimum number of vertices

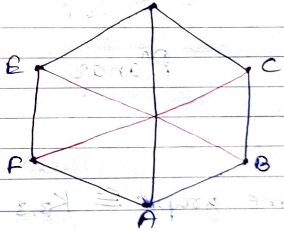
Every vertex in V_1 is connected to every vertex V_2 by an edge.

Complete Bipartite Graph :-

G(V, E) where $|V| = |V_1| + |V_2|$

$|V_1| = m$ $|V_2| = n$

$V_1 = \{A, C, E\}$ $V_2 = \{B, D, F\}$



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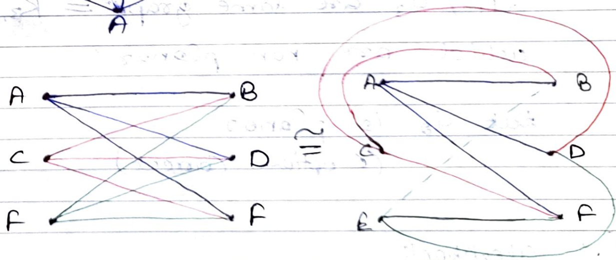


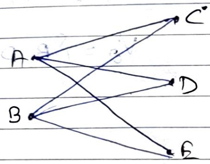
* $K_{3,3}$ is non planar graph with minimum number of edges.

~~others~~ in G(V, E)
we have $|V| = |V_1| + |V_2|$
$= m + n$

i) It is denoted by $K_{m,n}$

ii) $|E(K_{m,n})| = m \cdot n$

iii) $|V(K_{m,n})| = m+n$



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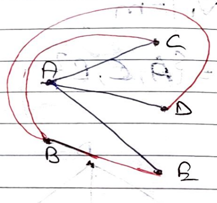


$K_{2,3}$ Planar

Q.1 workbook

G_1, G_3 are same graph ≅ $K_{3,3}$ which are non planar

But G_2 is planar
(C option Correct)

Q.2 Workbook

Ans - D) Based on Kurtowiskie's Theorem (Next Page)

(Not in GATE SYLLABUS) but Remember the statement.

Kurtowiskie's Theorem:-

A graph G(V, E) is planar iff it does not contain a subgraph which is homeomorphic to either K_5 (or) $K_{3,3}$
↓
(Read from internet)

Q.10 (Work Book) GATE:-2005

$|V| + |R| = |E| + 2$

$13 + |R| = 19 + 2$

$\boxed{|R| = 8}$

Same model Question asked in G:2021

Faces → Region

Q.11 (W.B) G:2005

Ans (A) correct option

Q.13 G:2007

$K_{3,3}$

$|V| = 3+3 = 6$

$|E| = 3 \cdot 3 = 9$

Option B correct

Q22 G. 2012

$|V| = 10$, $|E| = 15$

$|R| = ?$

$|V| + |R| = |E| + 2$

$10 + |R| = 15 + 2$

$|R| = 7$

6 Bounded Region — 1 Unbounded Region

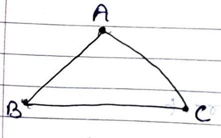
∴ 6 is the correct Answer

D option Correct

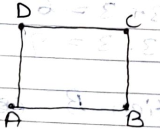
Cycle Graph

i) It is denoted by 'Cn' $(n \geq 3)$

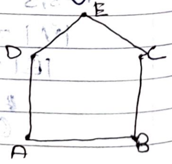
ii) $|E(C_n)| = |V(C_n)| =$ length of the cycle $= n$



C_3



C_4



* Regular Graph

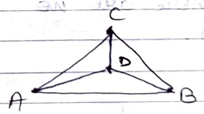
In K-regular graph degree of every vertix is 'K'.

→ Every cycle graph is 2-regular graph

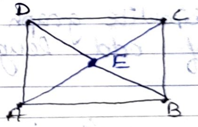
Wheel Graph

i) Denoted by W_n

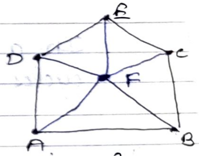
ii) W_n is obtained from 'C_{n-1}' by adding hub which is adjacent to all other vertices



W_4



W_5



W_6.

iii) $|E(W_n)| = 2(n-1)$

We know that $\sum_{i=1}^{n} deg(v_i) = 2|E|$

In W_n, there are $(n-1)$ vertices having degree '3' and hub has degree $(n-1)$

(n-1) vertices having degree '3'

Hub

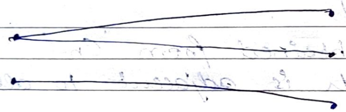
$$3(n-1) + 1(n-1) = 2|E|$$

$$\therefore \boxed{|E| = 2(n-1)}$$

Bipartite Graph

$G(V, E)$

V_1 V_2

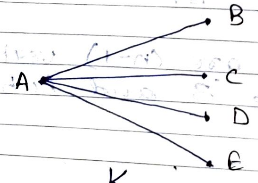


In a bipartite graph there are no cycles of odd length.

Star Graph (having n vertix)

i) $K_{1,n-1}$

ii) $E(K_{1,n-1}) = 1 \cdot (n-1) = n-1$

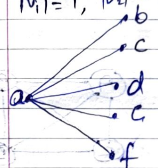


$K_{1,4}$

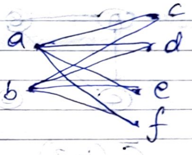
Q.51 (workbook)

$G(6 \quad)$

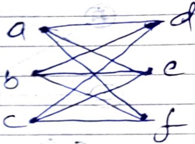
$|V_1| = 1, |V_2| = 5$



$|V_1| = 2 \quad |V_2| = 4$



$|V_1| = 3 \quad |V_2| = 3$



$\therefore E(K_{1,5}) = 5 \qquad E(K_{2,4}) = 8 \qquad E(K_{3,3}) = 9$

$G(V, \quad)$

$|V_1| = \frac{|V|}{2} = \frac{n}{2} \qquad |V_2| = \frac{|V|}{2} = \frac{n}{2}$

$\therefore$ Max. no. of edges $K_{\frac{n}{2}, \frac{n}{2}} = \frac{n}{2} \times \frac{n}{2}$

$$= \left\lfloor \frac{n^2}{4} \right\rfloor$$

for $n = 12$

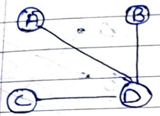
$$\left\lfloor \frac{12^2}{2} \right\rfloor$$

$= 36$

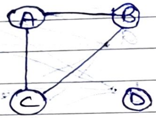
Complement of a Graph

Complement of a graph

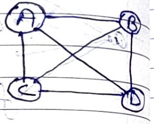
$G(V,E)$ $\bar{G}(V,E)$



$G(4,3)$



$\bar{G}(4,3)$



K_4

* $|E(G)| + |E(\bar{G})| = |E(K_n)|$

$= \frac{n(n-1)}{2}$

Q.52 workbook

A cycle graph with n vertices is isomorphic to its complement. The value of n is ______.

$C_n \cong \bar{C_n}$

$|E(C_n)| = |E(\bar{C_n})| = n$

$\underbrace{|E(G)|}_{\downarrow \; n} + \underbrace{|E(\bar{G})|}_{\downarrow \; n} = |E(K_n)|$

$n + n = \frac{n(n-1)}{2}$

$2n = \frac{n(n-1)}{2}$

$n = 5$

$G_1 \cong G_2$
↳ isomorphic sign

i) $|V(G_1)| = |V(G_2)|$

ii) $|E(G_1)| = |E(G_2)|$

Q.21) workbook

* How to verify given degree sequence is valid (or) not?

i) $\sum_{i=1}^{n} deg(v_i) = 2|E| =$ even

ii) No. of odd degree vertices are always even.

iii) If I have 'n' vertices, then following degree sequence is not allowed

$\underset{deg(v_1)}{n-1}, \underset{deg(v_2)}{n-1}, \ldots, 2, 1$

iv) If Graph has 'n' vertices then we cannot get a vertex whose degree is $> n-1$

i) & iii) are following the above 4 rules

∴ (ii) & (iv) are not Valid option D

1:14:30

Havel - Hakimi Result

i) $s, t_1, t_2, t_3, ----, t_s, d_1, d_2, ---- d_n$

ii) $t_1-1, t_2-1, t_3-1, ---, t_s-1, d_1, d_2, ---, d_n$

here seq (i) is descending order

seq (i) is graphic iff seq (ii) is graphic

↓

If degree sequence is valid sequence then it is called graphic

* Solving the previous question with Havel - Hakimi Result

i) $\boxed{7}$ 6 5 4 4 3 2 1

$\boxed{5}$ 4 3 3 2 1 0

$\boxed{3}$ 2 2 1 0 0

1 1 0 0 0

Since there are even no. of 1's therefore this sequence is valid

Steps to follow

1) First check the given sequence is in descending order, if it is not then arrange it in descending order.

2) Select the first no. ~~and~~ ~~&~~ Eg. if it is 7 then mark 7 no. next to it.

3) Drop down those marked no. with 1 subtracted from it.

7 6 5 4 3 2 2 1

5 4 4-1=3 2 1 1 0

4) Still if you can't able to identify that it is valid sequence or not then again you can apply Havel - Hakimi Result.

ii) $\boxed{6}$ 6 6 6 3 3 2 2

$\boxed{5}$ 5 5 2 2 1 2

~~4 4 1 1~~

$\boxed{5}$ 5 5 2 2 2 1 } Not valid sequence

$\boxed{4}$ 4 1 1 1

3 0 0 0

~~Not valid~~

III $\boxed{7}$ 6 6 4 4 3 2 2

↓ ↓ ↓ ↓ ↓ ↓ ↓

$\boxed{5}$ 5 3 3 2 1 1

↓ ↓ ↓ ↓ ↓

4 2 2 1 0 1

$\boxed{4}$ 2 2 1 1 0

↓ ↓ ↓ ↓

1 1 0 0

Even 1's therefore this sequence is valid.

IV $\boxed{8}$ 7 7 6 4 2 1 1

↓ ↓ ↓ ↓ ↓

$\cancel{6}$ $\cancel{6}$ $\cancel{5}$ $\cancel{3}$ $\cancel{1}$

~~If 8~~ ~~we~~ Degree 8 is not possible if total no. of vertices are 8

∴ this sequence not possible

I, III → Valid sequence

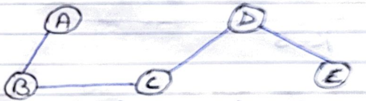
II, IV → Invalid sequence

29/4/21

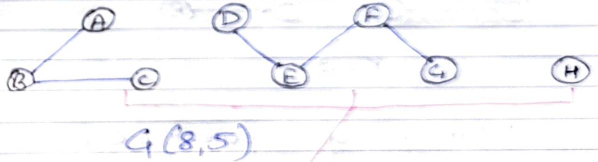
Lecture 15A

Connectivity

- An undirected graph is said to be connected if there exist a path between every pair of vertices.
- If the graph is not connected, then it contains connected components.



Connected Graph



This graph has '3' connected components

Q.4 Workbook

|V| = n

No. of components = K,

if we remove one vertex from G

III $\boxed{7}$ 6 6 4 4 3 2 2

↓ ↓ ↓ ↓ ↓ ↓ ↓

$\boxed{5}$ 5 3 3 2 1 1

↓ ↓ ↓ ↓ ↓

4 2 2 1 0 1

$\boxed{4}$ 2 2 1 1 0

↓ ↓ ↓ ↓

1 1 0 0

even 1's therefore this sequence is valid.

IV $\boxed{8}$ 7 7 6 4 2 1 1

$\cancel{6}$ $\cancel{6}$ $\cancel{5}$ $\cancel{3}$ $\cancel{1}$

~~If 8~~ Degree 8 is not possible if total no. of vertices are 8

∴ this sequence not possible

I, III → Valid sequence

II, IV → Invalid sequence

29/4/21

Lecture 15A

Connectivity

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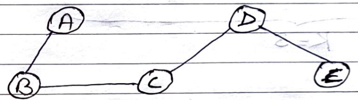
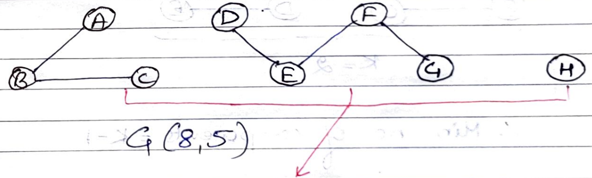


Connected Graph



This graph has '3' connected components

Q.4 workbook

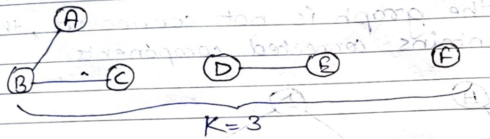
|V| = n

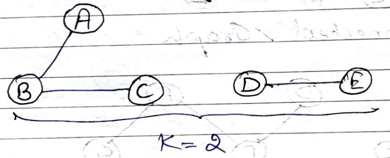
No. of components = K;

if we remove one vertex from 'G'

Case (i) :-
If the removable vertix is an isolated vertix then total no. of components after removing that vertix = K-1.

Ex



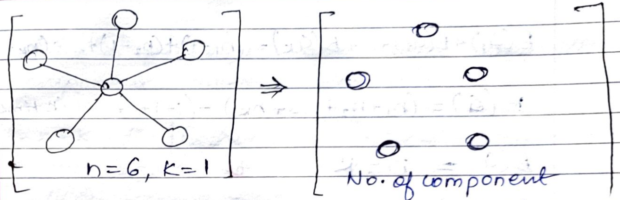


∴ Min no. of components = K-1

Case (ii)
If Graph has one component which is stargraph, and if we remove vertix of stargraph (which is connected to remaining vertices)

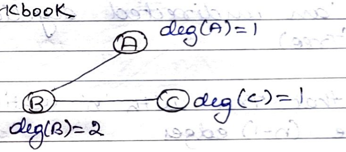
then we have max no. of components

= n-1



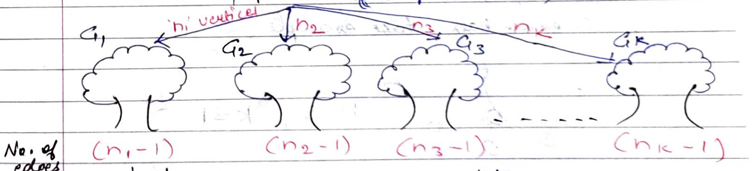
No. of component
= n-1
= 5

Q19 Workbook



Q26 (W.B)

$|V| = n$ (component 'k')



$|V| = n = n_1 + n_2 + \dots + n_k$

$|E(G_1)| = n_1 - 1$

$|E(G_2)| = n_2 - 1$

⋮

$|E(G_k)| = n_k - 1$

(Tree)

A forest is a acyclic graph, having (n-1) edges if it has 'n' vertices.

$E(G_1)+E(G_2)+\dots+E(G_k) = (n_1-1)+(n_2-1)+\dots+(n_k-1)$

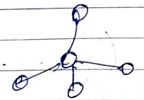
$E(G) = (n_1+n_2+\dots+n_k) - (1+1+1+\dots+1 \text{ (K times)})$

$E(G) = n - K$

Ans C

- Forest is an undirected acyclic graph (Tree)
- We know that Tree with 'n' vertices must have (n-1) edges
- Given that there 'K' connected components and every component is acyclic (Tree)

In exam hall take example



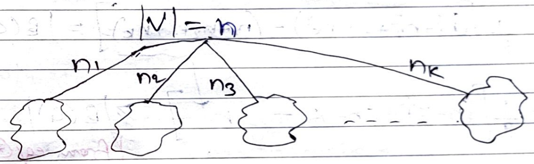
$n=5, \; K=1$

$|E(G)| = 4 = 5-1 = n-K$

Note:- Let G(V,E) be a graph with 'K' connected components then

$$n-K \le |E(G)| \le \frac{(n-K)(n-K+1)}{2}$$

where $|V| = n$



$|V| = n_1+n_2+n_3+\dots+n_K$

* If every component is tree

$\therefore (n_1-1)+(n_2-1)+(n_3-1)+\dots+(n_K-1) = (n-K)$

* sq. both sides

$\Rightarrow (n_1-1)^2+(n_2-1)^2+\dots+(n_K-1)^2 + (\text{Some extra terms}) = (n-K)^2$

$\Rightarrow (n_1^2-2n_1+1)+(n_2^2-2n_2+1)+\dots+(n_K^2-2n_K+1) \le (n-K)^2$

$\Rightarrow (n_1^2+n_2^2+\dots+n_K^2) - 2(n_1+n_2+\dots+n_K) + (\overbrace{1+1+\dots+1}^{\text{K times}}) \le (n-K)^2$

$\Rightarrow \boxed{(n_1^2+n_2^2+\dots+n_K^2) \le (n-K)^2 + 2n - K} \longrightarrow ⊛$

* If every component is ~~ty~~ complete graph.
(i.e max no. of edges)

⇒ $\frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \dots + \frac{n_K(n_K-1)}{2} = |E(G)|$

⇒ $\frac{n_1^2-n_1}{2} + \frac{n_2^2-n_2}{2} + \dots + \frac{n_K^2-n_K}{2} = |E(G)|$

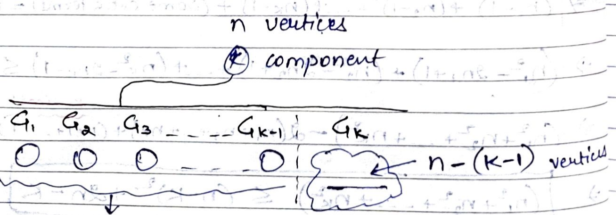
⇒ $\frac{1}{2}\left[(n_1^2+n_2^2+\dots+n_K^2) - \underbrace{(n_1+n_2+\dots+n_K)}_{}\right] = |E(G)|$

⇒ $\frac{1}{2}\left[(n-K)^2+2n-K - n\right] \geq |E(G)|$ [from eq ⊛]

⇒ $\frac{1}{2}\left[(n-K)^2+(n-K)\right] \geq |E(G)|$

⇒ $\boxed{\frac{1}{2}(n-K)(n-K+1) \geq |E(G)|}$

Case where max no. of edges are present



each component have only one vertex
(∴ ~~edge~~ No. of edge = 0)

$\frac{n(n-1)}{2}$ → No. of edges in complete graph

$0+0+0+\dots+0+\frac{(n-(K-1))(n-(K-1)-1)}{2} = |E(G)|$

$|E(G)| = \frac{(n-(K-1))(n-(K-1)-1)}{2}$

$|E(G)| = \frac{(n-K+1)(n-K)}{2}$

1:15:00

1:34

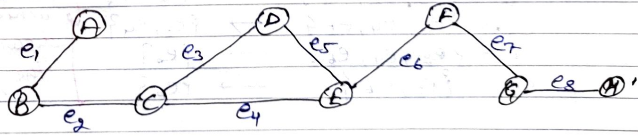
30/4/21

Cut Vertix (Articulation Point) :-

A vertix 'v' is called cut vertix, whose removal disconnect the graph into two (or) more components.

Cut Edge (Bridge) :-

An edge 'e' in 'G' is called cut edge if G-{e} result the graph as disconnected.



Cut Vertices :- C, E, G, F, B
Cut Edge :- <B,C>, <E,F>, <F,G>, <G,H> <A,B>

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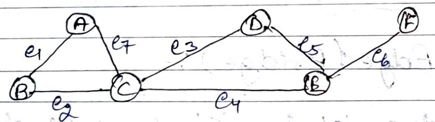
Cutset :-

Let G(V, E) be a graph.

Let C ⊆ E such that if we remove all edges which are in 'C' will result the graph as disconnected and no other proper subset of 'C' disconnects the graph then 'C' is called cutset.

i) C ⊆ E
ii) By removing all edges in 'C' the graph becomes disconnected.
iii) No proper subset of 'C' disconnects the graph



$C_1 = \{e_3, e_4\}$ ✓

$C_2 = \{e_3, e_2, e_7\} \longrightarrow \{e_2, e_7\}$

$C_3 = \{e_3, e_4, e_5\} \longrightarrow \{e_3, e_4\}$

$C_4 = \{e_2, e_4, e_6\} \longrightarrow \{e_6\}$

$C_5 = \{e_5, e_6\} \longrightarrow \{e_6\}$

Proper Subset which disconnects the graph

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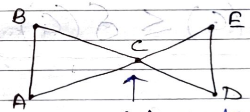
Properties of Connectivity

1) $n-K \leq E(G) \leq \frac{(n-K)(n-K+1)}{2}$

where $|V| = n$, 'K' components.

2) If the graph has cycle then cut edge is not a part of that cycle.

3) If Graph has cut vertix then it need not have cut edge.



cut vertex but there is no cutedge

4) If the graph has cutedge then it need not have cut vertix.

5) Vertix Connectivity:-

i) It is denoted by K(G)

ii) K(G) = Minimum no. of vertices are required to delete from the graph, inorder to convert the graph as dis-disconnected graph.

iii) If the graph has cut vertix then K(G)=1.

6) Edge Connectivity :-

i) ~~Deno~~ Denoted by $\lambda(G)$

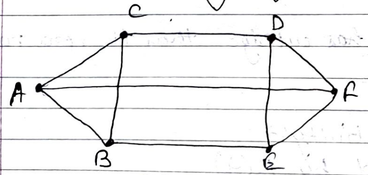
ii) $\lambda(G)$ = Minimum no. of edges are required to delete so as to convert graph as disconnected.

iii) If the graph has cut edge

$\lambda(G) = 1$

7) $\boxed{K(G) \leq \lambda(G) \leq \delta(G)}$

Q Find vertix connectivity, Edge connectivity for the following graph.



$\delta(G) = 3$

$K(G) \leq \lambda(G) \leq \delta(G)$

Since there is no cut-vertix and cut edge

∴ Vertix connectivity and edge connectivity is greater than 1.

$K(G) = 3$ {A, B, D}

$\lambda(G) = 3$ {<B,E><A,F><C,D>} (or)

{<E,F> <E,D> <D,F>}

Euler Path

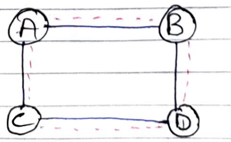
Every edge should be covered only one time and vertices can be covered multiple times.

Properties

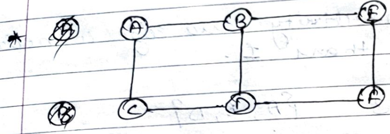
i) If the graph has exactly two vertices of odd degree then it has Euler path.

ii) Euler circuit will exist iff the graph has all vertices of even degree.

iii) If the graph has Euler circuit then it must contain Euler path.



A – B – C – D – A



→ B – E – F – D – C – A – B – D } Euler Path

→ D – F – E – B – A – C – D – B

Q: 16:00

Properties can be written as:-

1) A multigraph is having Euler path if it contains exactly two vertices of odd degree (or) all vertices of even degree.

2) If the graph is having Euler path then it is said to be Traversable.

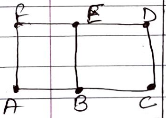
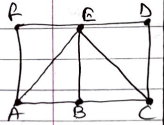
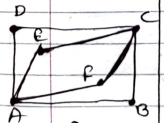
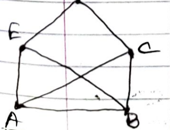
3) If the graph is having more than two vertices of odd degree then graph is not Traversable.

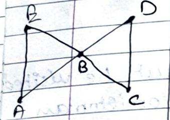
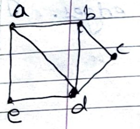
Hamiltonian Path

In Euler Path every edge must be visited exactly once where as in Hamiltonian Path every vertix must be visited exactly once and you may skip some edges.

Hamiltonian Circuit

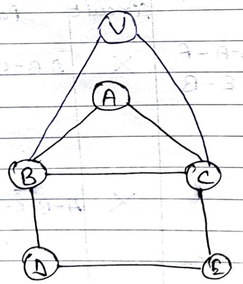
If the hamiltonian path is closed path (i.e Starting Vertix = Ending Vertix) then it is called Hamiltonian Circuit.

	Euler Path	Euler Circuit	Hamiltonian Path	Ham. Circuit
	E-D-C-B-A-F-E-B	✗	A-B-C-D-E-F-A	✓
	✗	✗	A-B-C-D-E-F-A	✓
	✓	✓	✗	✗
	✗	✗	A-B-C-D-E-A	✓

	Euler Path	Euler Circuit	Hamiltonian Path	Hamiltonian Circuit
	✓	✓	A-E-B-D-C	✗
	✓	✗	e-d-c-b-a-e	✓

Note:- If the graph has hamiltonian circuit then it is called hamiltonian.

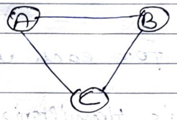
Q.17 W.B



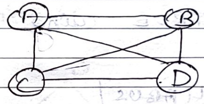
All vertices are even

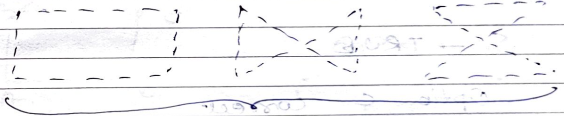
∴ Eulerian (Ans d)

Q.29



∴ Correct option may be (B) or (D)





$\frac{(n-1)!}{2} = \frac{(4-1)!}{2} = 3$

Option D ✓

Q.39 Ans C (Using Ore's Theorem)

(OUT OF GATE SYLLABUS)

~~Ore's Theorem~~

A simple graph with n vertices $(n \geq 3)$ is Hamiltonian if the sum of degrees of every pair of non-adjacent vertices is at least 'n'.

Q.T7

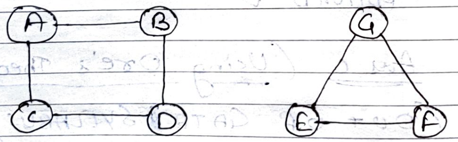
S_1 : If $d(V) \geq \frac{n}{2}$ for each vertex V in G, then G is Hamiltonian.
[d(V) is the degree of vertex V]

This statement is TRUE using Dirac's Theorem
[OUT OF GATE SYLLABUS]

S_2 → TRUE

Option C correct

Q.14



G(7,7)

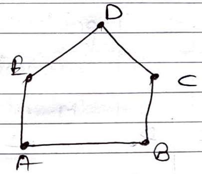
2-regular graph but Euler circuit does not exist

Option A becomes false if the graph is disconnected.

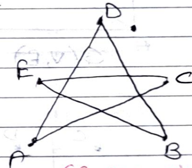
Option(B) → FALSE

In K_{90}, degree of every vertix is 89 which is odd. so Euler circuit does not exist.

Option (C) → TRUE

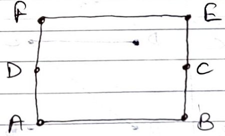


C_5

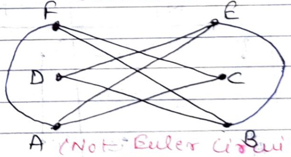


(Euler circuit)

$\overline{C_5}$ → odd



C_6



(Not Euler circuit)

$\overline{C_6}$ → ~~odd~~ even

In $\overline{C_{25}}$ every vertix is having even degree

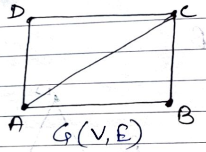
∴ $\overline{C_{25}}$ has Euler Circuit

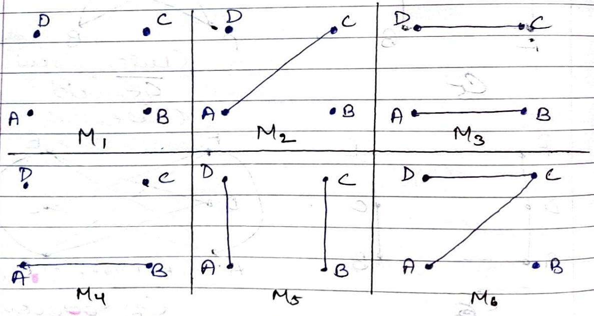
Matching

It must have same no. of vertices

i) M is sub graph of G

ii) $\deg(v) \leq 1$ in M $\forall\ v \in G$





Maximal Matching :- No more edge can't be added in subgraph

Maximum Matching :- Out of one which are maximal matching the one who has maximum no. of edges is called Maximum Matching.

	M_1	M_2	M_3	M_4	M_5	M_6
i) Matching	✓	✓	✓	✓	✓	✗
ii) Maximal Matching	✗	✓	✓	✗	✓	✗
iii) Maximum Matching	✗	✗	✓	✗	✓	✗
iv) Perfect Matching $\deg(v)=1$ in 'M'	✗	✗	✓	✗	✓	✗

Note:- No. of edges in maximum matching is called matching number.

(In the above example, Matching No is 2)

Note:- No. of perfect matching in $K_{2n} = \dfrac{2n!}{(n!)\,2^n}$

2n vertices are there

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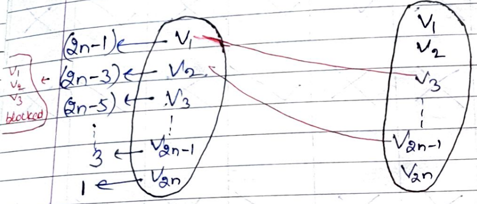
No. of perfect matching of $K_{2n} = 6$

here $2n = 6$



$= (2n-1)(2n-3)(2n-5)\ldots\ldots 3\cdot 1$

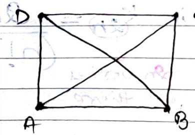
$= (6-1)(6-3)(6-5)$

$= 5 \times 3 \times 1$

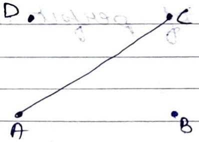
$= 15$

* No. of Perfect Matching in $K_{2n} = \frac{2n!}{(n!)2^n}$

Proof:



$K_4 = K_{2n}$



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Vertix 'A' can match in 3 ways

Vertix 'B' can match in 1 ways

If there are 2n vertices, then

1^{st} vertix match in $(2n-1)$ ways

2^{nd} vertix match in $(2n-3)$ ways

3^{rd} vertix match in $(2n-5)$ ways

⋮

last vertix .... 1 way

Total no. of ways $= (2n-1)(2n-3)(2n-5)\ldots\ldots 3\cdot 1$

$$= \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\ldots\ldots 4\cdot 3\cdot 2\cdot 1}{(2n)(2n-2)(2n-4)\ldots\ldots 4\cdot 2}$$

$$= \frac{2n!}{2^n\ n!}$$

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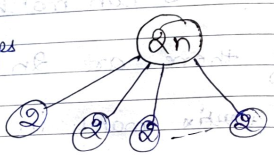
Another way of proving
Divide & Distribution.

$2n \rightarrow \{x_1, x_2, x_3, \ldots, x_{2n}\}$

(- -)
(- -)
(- -)
|
(- -)

n groups



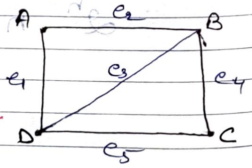
$$= \frac{(2n)!}{\underbrace{(2!)(2!)(2!)\ldots(2!)}_{n \text{ times}}\; n!}$$

$$\frac{(2n)!}{2^n \cdot n!}$$

Line Covering

$\downarrow$ Edge $\quad \downarrow$ Covering

Edge represented by < >



$S_1 = \{<A,B> <C,D>\}$ ✓

$S_2 = \{<A,D> <B,C>\}$ ✓

$S_3 = \{<A,B> <B,D> <C,D>\}$ ✓

$S_4 = \{<A,B> <B,C> <B,D>\}$ ✓

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$S_5 = \{<A,B> <B,D>\}$ ✗ (C vertex is missing)

* Every vertex should appear, then only it is line covering.

* If after removing one edge the subset is loosing property of line covering then it is called Minimal line Covering.

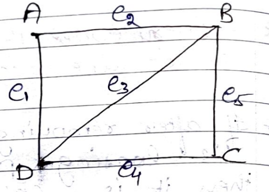
* From minimal L.C, the set which has minimum no. of edge, then that set is called minimum L.C.

	L.C	Minimal L.C	Minimum L.C
S_1	✓	✓	✓
S_2	✓	✓	✓
S_3	✓	✗	✗
S_4	✓	✓	✗
S_5	✗	✗	✗

* Line Covering Number = No. of edges in Minimum line covering

In this example it is 2.

Vertix Covering



V= {A, B, C, D}

K1 = {B, D} ✓

K2 = {A, B, C} ✓

K3 = {A, D, C} ✓

K4 = {A, C} ✗

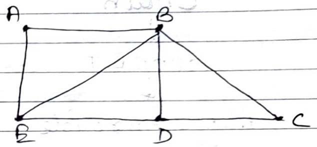
K5 = {A, B, C, D} ✓

	V. C	Minimal	Minimum
K1	✓	✓	✓
K2	✓	✓	✗
K3	✓	✓	✗
K4	✗	✗	✗
K5	✓	✗	✗

Vertix Covering Number = No. of vertices in minimum vertex covering

(In this example it is 2)

Independent Set (No common edge between given two vertices)



S1 = {A, D}

S2 = {B}

S3 = {E, C}

S4 = {A, C}

S5 = {B, E}

	Independent set	Maximal	Maximum
S1	✓	✓	✓
S2	✓	✓	✗
S3	✓	✓	✓
S4	✓	✓	✓
S5	✗	✗	✗

* Independent set no. = No. of vertices in Maximum Independent Set

{In this example, it is 2}

1:11:16

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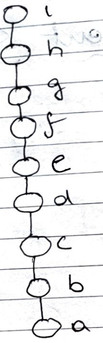
Q.14

Chain
↓
TOSET
↓
~~Always~~ Electric Pole



Independent set

{a, c, e, g, i}

{b, d, f, h}

{a, d, g, i}

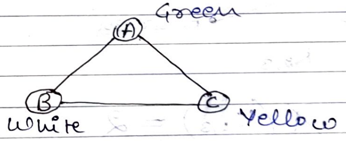
{b, e, h} → 3 ✓ - (Ans)

* Line Covering No. + Matching No. = |V|

* Vertex Covering No + Independent set Number = |V|

Coloring

Minimum no. of colors are required to color the graph such that no two adjacent vertices have the same color is called chromatic number of graph 'G' and it is denoted by $\chi(G)$.
This process is called vertex coloring.
We can apply the same process for coloring edges then it is called edge coloring.
(Refer GATE 2020)



$\chi(G) = 3$

→ If the graph uses atmost n-colors to color the graph then graph is n-colorable.

★★★★★

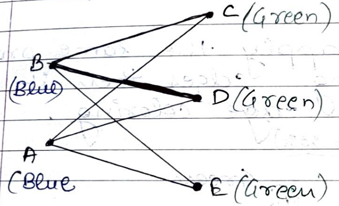
* Every planar graph is 4-colourable. (Four color Theorem)

$\chi(G) \leq 4$

Welch Powel's Algorithm

Arrange vertices in the decreasing order of their degrees and then color those vertices in that order.

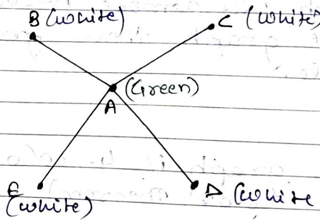
Q $\chi(G) = ?$



$K_{2,3}$

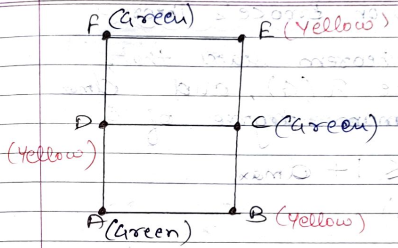
$\therefore \chi(K_{2,3}) = 2$

$\boxed{\chi(K_{m,n}) = 2}$

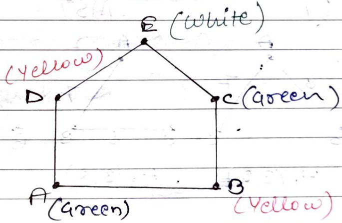


$\chi(K_{1,4}) = 2$

$\chi(K_{1,n-1}) = 2$



$\chi(\underset{\text{even}}{C_6}) = 2$



$\chi(\underset{\text{odd}}{C_5}) = 3$

Q.7 W.B → Option C (4)

Q.16 W.B → Option B (3)

Q.18

$\chi(G) = 2$

(If graph does not have odd length cycle then it becomes ~~complete~~ bipartite graph.)

$\therefore \chi(K_{m,n}) = 2$

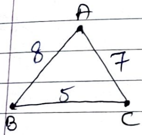
Question based on Brooke's Theorem

Q.36: Brooke's theorem states that, if 'c' is the $X(G)$, and d_{max} is the maximum degree of G then

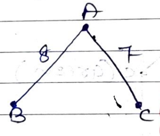
$$C \le 1 + d_{max}$$

(Ans - d)

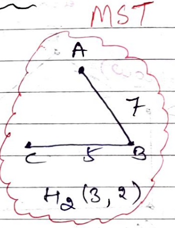
Spanning Tree



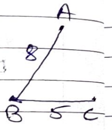
G(3,3)



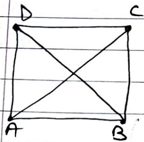
H₁ (3,2)



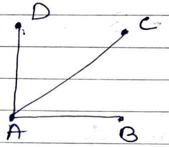
H₂ (3,2)



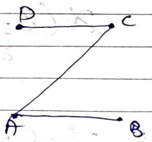
H₃ (3,2)



G(4,6)



H₁ (4,3)



H₂ (4,3)

1) Spanning tree must have same number of vertices

2) Spanning tree must be acyclic

3) Spanning tree must be connected (i.e b/w any two vertices we can find a path)

Properties of Spanning Tree

i) G(V, E), No. of edges in spanning Tree $= |V| - 1$

ii) To construct spanning Tree of G(V, E) we have to remove $\underbrace{(E - V+1)}_{\text{Circuit Rank}}$ no. of edges from the graph.

iii) If G(V, E) is a complete graph No. of spanning Tree $= V^{V-2}$

iv) In $K_{m,n}$ No. of spanning Tree $= m^{n-1} \times n^{m-1}$

$K_{3,3}$ .... $= 3^{3-1} \times 3^{3-1} = 81$

v) MST
- Kruskal's Algo
- Prim's Algo

Algorithm subject (Greedy Method)

vi) Every spanning Tree is maximally acyclic i.e if we add one more edge to the spanning Tree then it forms a cycle.

vii) Every spanning tree is minimally connected (After removing one edge graph becomes disconne

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Question based on Brooke's Theorem

Q.36: Brooke's theorem states that, if 'C' is the $\chi(G)$, and d_{max} is the maximum degree of G then

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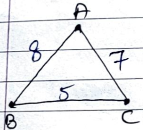
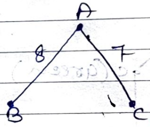
(Ans - d)

Spanning Tree

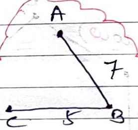


G(3,3)

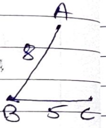


H₁(3,2)

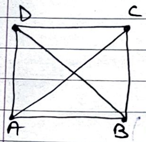
MST



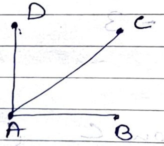
H₂(3,2)



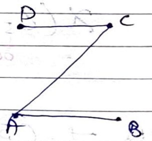
H₃(3,2)



G(4,6)



H₁(4,3)



H₂(4,3)

1) Spanning tree must have same number of vertices

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No. of spanning Tree $= V^{V-2}$

iv) In $K_{m,n}$
No. of spanning Tree $= m^{n-1} \times n^{m-1}$

- $K_{3,3}$ .... $= 3^{3-1} \times 3^{3-1} = 81$

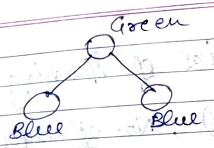
v) MST
- Kruskal's Algo
- Prim's Algo

} Algorithm subject (Greedy Method)

vi) Every spanning Tree is maximally acyclic i.e if we add one more edge to the spanning Tree then it forms a cycle.

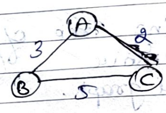
vii) Every spanning tree is minimally connected (After removing one edge graph becomes disconnected)

Q.38 1.) TRUE

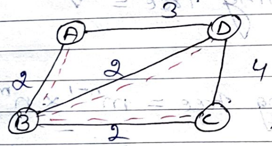


2) TRUE

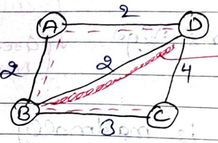
3)



∴ Weight of Minimum Cost Spanning Tree (MST)
= 2+3
= 5



Weight of MST = 2+2+2
= 6



→ Can't be taken in MST as spanning tree will become cyclic

Weight of MST = 2+2+3 = 7

Statement (3) **FALSE**

Option (a) is TRUE

Isomorphism of Graphs

Two graphs G and G' are said to be Isomorphic if there is a function

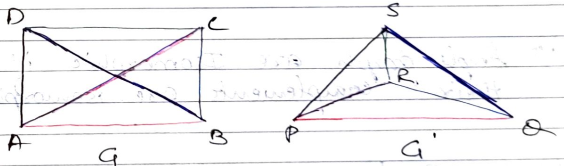
$$f : V(G) \longrightarrow V(G')$$ such that

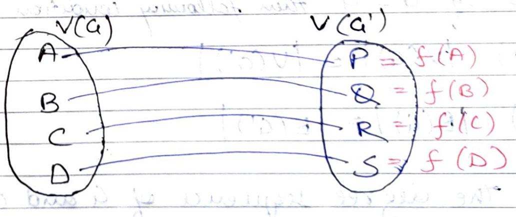
i) f is a bijective (one-one and onto)

ii) for each pair of vertices u and v of G,

$<u, v> \in E(G)$ iff $<f(u), f(v)> \in G$

$<f(u), f(v)> \in G$





<u, v>	<f(u), f(v)>
<A,B>	<P,Q>
<A,C>	<P,R>
<A,D>	<P,S>
<B,C>	<Q,R>
<B,D>	<Q,S>
<C,D>	<R,S>

Note: $G \cong G'$ iff

Adjacency Matrix of G $\cong$ Adjacency Matrix of G'

[Under 'f']

Note: Simple graph are Isomorphic if their complements are isomorphic.

Note: If $G \cong G'$ then following condition must hold

i) $|V(G)| = |V(G')|$

ii) $|E(G)| = |E(G')|$

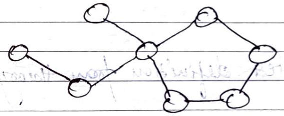
iii) The degree sequence of G and G' are same

~~iv) The number of simple circuit of a given~~

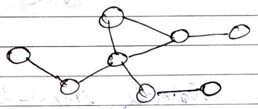
iv) The number of simple cycle circuit of a given length must be same in both graph.

1:21:37

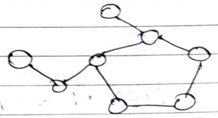
Q Which of the following graph is Isomorphic to?



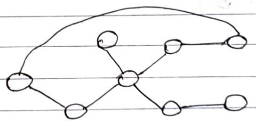
a)



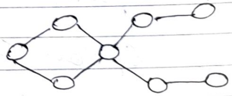
b)



c)



d)



Ans C ~~from~~
From the previous four properties option C is satisfying all of them.

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Q.12 W.B

Option D

Q.24 Option C

$^{8}C_4 \times 3! = 90$

Q.30:- Read Diameter definition from theory book

Q.31 A

Q.32 D

Q.33 Read Theory book for unilateral, strongly connected.

Q.34 Bipartite (Option B)

Q.35 Option C

Q.37 Out of syllabus

Q.40 A

Q.41 D

Q.43 D

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Q.44 3

Q.45 3

Q.46 4.

Q.47 $|E(G)| + |E(\bar{G})| = |E(K_n)|$

Q.48 $(|V| + |R| = |E| + 2)$

Q.49 $^{7}C_5 - 6$

Q.50 506

Q.51 : 36

Q.52 : 5

Q.53: 24

Q.54 Data Structure (199)

Q.55 $\chi(G) \leq$ (Four Color Theorem)

Q.56 18

Q.57 16

Q.58 3

Q.27) $\delta \le \frac{2|E|}{|V|} \le \Delta$

$\delta \le \frac{2|E|}{|V|}$

$2|E| \geq n\cdot\delta$

$|E| \geq \frac{n\cdot\delta}{2}$

$|V|+|R| = |E|+2$

$n+|R| \geq \frac{n\cdot\delta}{2}+2$

$|R| \geq \frac{n\delta}{2}+2-n$

$|R| \geq \frac{3n}{2}+2-n \quad [\because \delta \geq 3]$

$$\boxed{|R| \geq \frac{n}{2}+2}$$

Q.28) $G \cong \bar{G} \quad (E(G) = E(\bar{G}))$

$|E(G)| + |E(\bar{G})| = |E(K_n)|$

$2|E(G)| = \frac{n(n-1)}{2}$

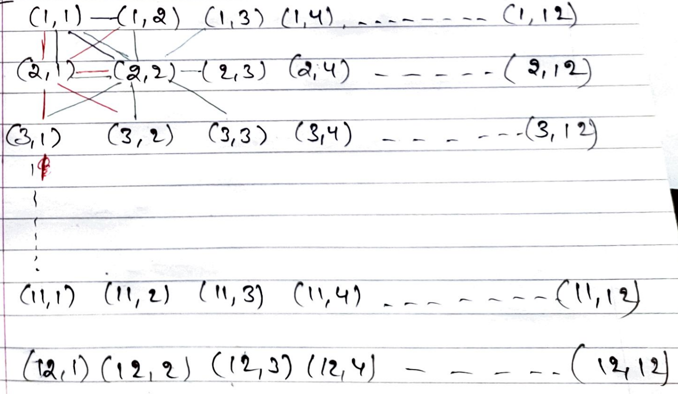
$|E(G)| = \frac{n(n-1)}{4}$

i.e $\frac{n}{4}$ or $\frac{n-1}{4}$

$n \cong 0 \pmod 4$ or $n-1 \cong 0 \pmod 4$

$n \cong 1 \pmod 4$

Q.50



(1,1) (1,2) (1,3) (1,4) .... (1,12)

(2,1) (2,2) (2,3) (2,4) ...... (2,12)

(3,1) (3,2) (3,3) (3,4) ...... (3,12)

(11,1) (11,2) (11,3) (11,4) .... (11,12)

(12,1) (12,2) (12,3) (12,4) ...... (12,12)

i) 4 corner vertices and each degree 3.

ii) 40 side vertices whose degree 5

iii) 100 interior vertices of degree 8

$$\sum_{i=1}^{n} deg(v_i) = 2|E|$$

$$4(3) + 40(5) + 100(8) = 2|E|$$

$$1012 = 2|E|$$

$$\therefore |E| = \frac{1012}{2} = 506$$

1/5/21

Lecture 16B

Logic

Proposition Logic	First Order Logic
1) Connectivities	1) Sentences with Quantifier ∀ : for all ∃ : There exist
2) Tautology, Contradiction Contingency, Satisfiability	
3) Tautological Implications valid of an argument	
4) Rules of Inference	
5) Quine's Method	
6) CP Rule (Conditional Proof)	

Proposition:-

i) Sun Rises in the East TRUE

ii) $5+8=10$ FALSE

iii) He is an Intelligent!

iv) $x+2=8$

v) How are you?

vi) wow, She is Beautiful!

Not Proposition Logic

Atomic

p: Delhi is the Capital of India

q: $2+3=4$.

Compound

p and q : $p \wedge q$

p or q : $p \vee q$

If p then q : $p \rightarrow q$

p iff q : $p \leftrightarrow q$

not p : $\sim p$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \leftrightarrow q$
T	T	T
T	F	T
F	T	T
F	F	T

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	T

$\therefore \sim(\sim p) \equiv p$

18:35

2/5/19

Contingency = neither Tautology nor contradiction

P	Q	Tautology	contradiction	Contingency	satisfibility
T	T	T	F	T	
T	F	T	F	F	
F	T	T	F	F	T
F	F	T	F	T	

Atleast one True value. (Satisfibility)

* Every ~~Fatal~~ Tautology is Satisfiability.

* ~~Contio~~ Every contingency is satisfiobility

P	~P	P∨~P	P∧~P	P→~P
T	F	T	F	F
F	T	T	F	T

Tautology Satisfiability

Contradiction

Contingence Satisfibility

Tautological Implication

P Q

P tautologically implies Q means P → Q is a Tautology.

i) $\underbrace{\underset{T}{P} \longrightarrow Q}_{T}$ is a Tautology

By assuming L.H.S is 'T' we have to show R.H.S is 'T'

ii) $\underbrace{\underset{F}{P} \rightarrow \underset{F}{Q}}_{F}$ is a Tautology

By assuming R.H.S is 'F' we have to prove that L.H.S is F.

Valid Argument

i) $\dfrac{P_1, P_2, \ldots P_n}{\therefore Q}$ (One way of representing)

$\boxed{OR}$

ii) $\{P_1 \land P_2 \land P_3 \land \ldots \land P_n\} \rightarrow Q$ is a Tautology.

OR

2/5/19

P	Q	Tautology	contradiction	Contingency = neither Tautology nor contradiction	satisfibility
T	T	T	F	T	
T	F	T	F	F	
F	T	T	F	F	T
F	F	T	F	T	

Atleast one True value. (Satisfibility)

* Every ~~Fatat~~ Tautology is Satisfiability.

* ~~Contio~~ Every contingence is satisfiability

P	~P	P∨~P	P∧~P	P→~P
T	F	T	F	F
F	T	T	F	T

Tautology
Satisfibility

Contradiction

Contingence
Satisfibility

Tautological Implication

P Q

P tautologically implies Q means
$P \to Q$ is a Tautology

i) $\underbrace{\underset{T}{P} \longrightarrow Q}_{T}$ is a Tautology

By assuming L.H.S is 'T'
we have to show R.H.S is 'T'

ii) $\underbrace{\underset{F}{P} \longrightarrow \underset{F}{Q}}_{F}$ is a Tautology

By assuming R.H.S is 'F'
we have ~~to~~ prove that L.H.S is F.

Valid Argument

i) $\dfrac{P_1, P_2, \ldots\ldots P_n}{\therefore Q}$ (One way of representing)

OR

ii) $\{P_1 \wedge P_2 \wedge P_3 \wedge \ldots\ldots \wedge P_n\} \to Q$ is a Tautology.

OR

iii) P_1
P_2
$\vdots$
P_n
$\overline{\therefore Q}$

Rules Of Inferences

i) Simplification Rule

$(P\wedge Q) \Rightarrow P$. (∵ Read as $(P\wedge Q)$ tautologically implies P)

Tautological Implication sign

i.e $\underbrace{(\overset{T}{P}\wedge \overset{T}{Q})}_{T} \to \underbrace{P}_{T}$ is $\boxed{T}$ → Tautology

$P\wedge Q \Rightarrow P$

$\underbrace{\underbrace{(\overset{T}{P}\wedge \overset{T}{Q})}_{T} \to \underbrace{Q}_{T}}_{T}$ is T

2) Addition Rule

$P \Rightarrow (P\vee Q)$

$\underbrace{\underbrace{P}_{T} \to \underbrace{(\overset{T}{P}\vee \overset{T/F}{Q})}_{T}}_{T}$ is T

$Q \Rightarrow (P\vee Q)$

$\underbrace{\underbrace{Q}_{T} \to \underbrace{(\overset{T/F}{P}\vee \overset{T}{Q})}_{T}}_{T}$ is T

∴ It is valid.

6) $\sim(P\to Q) \Rightarrow \sim P$ is valid or not

If $\underbrace{\underbrace{\sim\overset{F}{(\underset{T}{P}\to \underset{F}{Q})}}_{T} \to \underbrace{\sim P}_{F}}_{F}$

∴ It is invalid

#) $\sim(P\to Q) \Rightarrow \sim Q$ is it valid (or) not

If $\underbrace{\underbrace{\sim\underbrace{(\overset{T}{P}\to \overset{F}{Q})}_{F}}_{T} \to \underbrace{\sim \overset{F}{Q}}_{T}}_{T}$

∴ It is valid

⑦ Disjunctive Syllogism ★★★★★

$$\{(P \vee Q) \wedge \sim P\} \Rightarrow Q$$

If $\underbrace{\{\underbrace{\underbrace{(\overset{F}{P} \vee \overset{T}{Q})}_{T} \wedge \underbrace{\sim \overset{F}{P}}_{T}}_{T}\} \rightarrow \underbrace{Q}_{T}}_{T}$ is 'T' then it is valid

$P \vee Q$
$\sim P$
$\therefore Q$

Example
Pen is in left pocket or Right pocket.
Immediately after that we are saying Pen is not in left pocket.
So obviously it will in Right pocket (i.e Q)

⑧ Conjunctive Syllogism

$$\underbrace{\underbrace{\{\overset{T}{\sim}\overset{F}{(\underset{T}{P} \wedge \underset{F}{Q})} \wedge \overset{T}{P}\}}_{T} \rightarrow \underbrace{\sim \overset{F}{Q}}_{T}}_{T}$$

~~{(P∨Q)}~~

$\{\sim(P \wedge Q) \wedge P\} \Rightarrow \sim Q$ is

$\{\sim(P \wedge Q) \wedge P\} \rightarrow \sim Q$ is 'T'

9) Modus Pones ★★★★★

$$\{(P \rightarrow Q) \wedge P\} \Rightarrow Q$$

If $\underbrace{\{\underbrace{\overset{T}{(\underset{T}{P} \rightarrow \underset{T}{Q})} \wedge \overset{T}{\underset{T}{P}}}_{T}\} \rightarrow \underbrace{Q}_{T}}_{T}$ is 'T' then it is valid.

$P \rightarrow Q$
P
$\therefore Q$

Another way of representing

10) Modus Tollens

$$\{(P \rightarrow Q) \wedge \sim Q\} \Rightarrow \sim P$$

Modus Pollens

$\{(P\to Q)\land\sim Q\}\to\sim P$

If

$$\{\underbrace{\overset{T}{(\underset{F}{P}\to\underset{F}{Q})}\land\overset{T}{\sim\overset{F}{Q}}}_{T}\}\to\underset{T}{\overset{F}{\sim P}}$$

is 'T' then it is valid

$\underbrace{\qquad\qquad}_{T}$

$$\begin{array}{l}\therefore\ P\to Q\\ \quad\ \sim Q\\ \hline \therefore \sim P\end{array}$$

11) Transitivity (Hypothetical Syllogism)

$\{(P\to Q)\land(Q\to R)\}\Rightarrow\{P\to R\}$

12) Dilema

$\{((P\lor Q)\land(P\to R)\land(Q\to R))\}\Rightarrow R$

R R

(By Modus Pones)

$\underbrace{R\lor R}_{T}\Rightarrow\underbrace{R}_{T}$

13) Constructive Dilema

$\{((P\lor Q)\land(P\to R)\land(Q\to S))\}\Rightarrow(R\lor S)$

S R

(By Modus Pones)

$$\underbrace{\underbrace{R\lor S}_{T}\Rightarrow\underbrace{(R\lor S)}_{T}}_{T}$$

14) Distructive Dilema

$\{((P\to R)\land(Q\to S)\land(\sim R\lor\sim S))\}$

~P ~Q

$\Rightarrow\sim P\lor\sim Q$

$\sim P\lor\sim Q$

Rules to solve Questions

1) $P \to Q$

P

$\therefore Q$

2) $P \to Q$

$\sim Q$

$\therefore \sim P$

3) $P \to Q$

$Q \to R$

$\therefore P \to R$

4) $(P \vee Q)$

$\sim P$

$\therefore Q$

$P \vee Q$

$\sim Q$

$\therefore P$

5) $P \to Q \cong \sim P \vee Q$

6) $P \to Q \cong \sim Q \to \sim P$

7) $\sim(P \vee Q) \cong \sim P \wedge \sim Q$

$\sim(P \wedge Q) \cong \sim P \vee \sim Q$

(Demorgan's Laws)

8) Converse of $(p \to q) = q \to p$

Inverse of $(p \to q) = \sim p \to \sim q$

Contrapositive of $(p \to q) = \sim q \to \sim p$

In general $p \to q \cong \sim q \to \sim p$

Duality Concept :-

$\vee$ Replaced $\wedge$

$\wedge$ Replaced $\vee$

Dual of $P \vee Q = P \wedge Q$

Precedence in Descending Order

$\sim, \wedge, \vee, \to, \leftrightarrow$

Q Consider the following conditional statement:

P: If the flood destroys my house or the fire destroys my house, then my insurance company will pay me.

which of the following is equivalent to p?

d) If my insurance company does not pay me then flood does not destroys my house and fire does not destroys my house

$$A \to B \cong \sim B \to \sim A$$

or

$$A \to B = \sim A \vee B$$

Rules to solve Questions

1) $P \rightarrow Q$
P
$\therefore Q$

2) $P \rightarrow Q$
$\sim Q$
$\therefore \sim P$

3) $P \rightarrow Q$
$Q \rightarrow R$
$\therefore P \rightarrow R$

4) $(P \vee Q)$
$\sim P$
$\therefore Q$

$P \vee Q$
$\sim Q$
$\therefore P$

5) $P \rightarrow Q \cong \sim P \vee Q$

6) $P \rightarrow Q \cong \sim Q \rightarrow \sim P$

7) $\sim (P \vee Q) \cong \sim P \wedge \sim Q$

$\sim (P \wedge Q) \cong \sim P \vee \sim Q$

(Demorgan's Law)

8) Converse of $(p \rightarrow q) = q \rightarrow p$

Inverse of $(p \rightarrow q) = \sim p \rightarrow \sim q$

Contrapositive of $(p \rightarrow q) = \sim q \rightarrow \sim p$

In general $p \rightarrow q \cong \sim q \rightarrow \sim p$

Duality Concept :-

$\vee$ Replaced $\wedge$

$\wedge$ Replaced $\vee$

Dual of $P \vee Q = P \wedge Q$

Precedence in Descending Order

$\sim, \wedge, \vee, \rightarrow, \leftrightarrow$

Q Consider the following conditional statement:

P: If the flood destroys my house or the fire destroys my house, then my insurance company will pay me.

Which of the following is equivalent to p?

d) If my insurance company does not pay me then flood does not destroys my house and fire does not destroys my house

$A \rightarrow B \cong \sim B \rightarrow \sim A$

or

$A \rightarrow B = \sim A \vee B$

$$\underbrace{(a \vee b)}_{p} \to \underbrace{i}_{q} \equiv \sim i \to \sim(a \vee b)$$

$$\equiv \sim q \to \sim p$$

$$\equiv \sim i \to (\sim a \wedge \sim b)$$

Option - D Correct

2/5/21

Lecture 17A

Q Consider the following arguments:

$S_1 : \{ r \to (q \to p), \sim p \} \Rightarrow (-r \vee \sim q)$

$S_2 : \{ (p \to q) \wedge (q \to r), (\sim q \wedge r) \} \Rightarrow p$

[, means $\wedge$ (and)]

which of the following is TRUE?

a) Only S_1 is valid

b) Only S_2 is valid

c) Both S_1 and S_2 are valid

d) Both S_1 and S_2 are invalid

1.) $r \to (q \to p) \equiv \sim r \vee (\sim q \vee p)$

($\because a \to b \equiv \sim a \vee b$)

$\equiv (-r \vee \sim q) \vee p$ → ①

$= r \to (q \to p) \wedge \sim p$

$= (\sim r \vee \sim q) \vee \underbrace{p \wedge \sim p}_{F}$

$= (\sim r \vee \sim q) \vee F$

$= \sim r \vee \sim q$

2.) $\{ (p \to q) \wedge (q \to r) \wedge (\sim q \wedge r) \} \Rightarrow p$

$$\{ \underbrace{\underbrace{(\overset{T/F}{p} \to \overset{T}{r})}_{T} \wedge (\underbrace{\overset{T}{\sim q} \wedge \overset{T}{r}}_{T})}_{T} \} \Rightarrow \underbrace{p}_{T/F}$$

∴ It is Invalid

(as p is taking 2 truth values)

Q The statement formula

$\{(a \vee b) \wedge (\sim a \vee c) \wedge \sim(b \vee c)\}$ is

a) a tautology

b) a contradiction

c) a contingency

d) None of these

$$\{\underbrace{\overbrace{(a \vee b) \wedge (\sim a \vee c)}^{(b \vee c)}}_{a \rightarrow c} \wedge \sim(b \vee c)\}$$

$$(b \vee c) \wedge \sim(b \vee c) \equiv f$$

∴ B is correct Answer

Quine's Method :-

Consider w.f.f (Well formed formula i.e given equation in ques), assume that w.f.f contain a variable 'A'.

Case (i)

Replace 'A' by True, if (w.f.f is TRUE) and Replace 'B' by False if (w.f.f is TRUE) then w.f.f is Tautology.

(i.e Replace 'A' by TRUE & FALSE and if in both case the value of w.f.f is True then it is Tautology)

Case (ii)

Replace 'A' by TRUE if (w.f.f is False) and Replace 'A' by FALSE if (w.f.f is false) then w.f.f is contradiction.

Case (iii)

Replace 'A' by TRUE, if w.f.f is TRUE and Replace 'A' by FALSE if w.f.f is ~~false~~ FALSE, then w.f.f is called contingence.

(If w.f.f (A by TRUE) is contingence (or) w.f.f (A by FALSE) is contingence, then w.f.f is called contingence)

Previous question by Quine's Method

Case(i) Replace 'a' by 'T'

$$(\underset{T}{a} \vee b) \wedge (\underset{F}{\overset{T}{\sim a}} \vee c) \wedge \sim(b \vee c)$$

$$\underbrace{T \wedge C}_{\downarrow} \wedge (\sim b \wedge \sim c)$$

$$= C \wedge (\sim b \wedge \sim c)$$

$$= (c \wedge \sim c) \wedge \sim b$$

$$= F \wedge \sim b$$

$$= F$$

Replace 'a' by F

$\underbrace{(\overset{F}{a} \vee b)}_{\downarrow} \wedge \overset{T}{(\sim \overset{F}{a} \vee c)} \wedge \sim(b \vee c)$

$\underbrace{b \wedge T}_{\downarrow} \wedge (\sim b \wedge \sim c)$

$= b \wedge (\sim b \wedge \sim c)$

$\equiv \underbrace{(b \wedge \sim b)}_{\downarrow} \wedge \sim c$

$= F \wedge c \equiv F$

Q Consider the following statements:

$S_1: \underbrace{(\underbrace{(\overset{T}{a} \vee \overset{T}{b})}_{T} \rightarrow \overset{F}{c})}_{F} \Rightarrow \overbrace{\underbrace{(\overset{T}{a} \wedge \overset{T}{b})}_{T} \rightarrow \underbrace{c}_{F}}^{F}$

$S_2: ((a \wedge b) \rightarrow c) \Rightarrow (a \vee b) \rightarrow c$

Which of the following is TRUE?

a) Only S_1 is valid
b) Only S_2 is valid
c) Both S_1 and S_2 are valid
d) Both S_1 and S_2 are invalid.

$S_1: \underbrace{(\underbrace{(\overset{T}{a} \vee \overset{T}{b})}_{T} \rightarrow \overset{F}{c})}_{F} \nRightarrow \overbrace{\underbrace{(\overset{T}{a} \wedge \overset{T}{b})}_{T} \longrightarrow \underbrace{c}_{F}}^{F}$

$\therefore$ L.H.S (F) $\longrightarrow$ R.H.S (F) $\equiv$ T

$S_2 \; \underbrace{(\underbrace{(\overset{T}{a} \wedge \overset{F}{b})}_{F} \longrightarrow \underbrace{c}_{F})}_{T} \Rightarrow \underbrace{\underbrace{(\overset{T}{a} \vee \overset{F}{b})}_{T} \rightarrow \underset{F}{c}}_{F}$

L.H.S (T) $\rightarrow$ R.H.S (F) $\equiv$ F

$\therefore$ Only S_1 is valid (Option A)

Q Consider the following statement formulae.

$S_1: \{(\sim p \rightarrow (q \rightarrow \sim w)) \wedge (\sim s \rightarrow q) \wedge \sim t \wedge (\sim p \vee t)\} \rightarrow (w \rightarrow s)$

$S_2: \{(q \rightarrow t) \wedge (s \rightarrow r) \wedge (\sim q \rightarrow s)\} \rightarrow (\sim t \rightarrow r)$

Which of the following is TRUE

a) Only S_1 is valid
b) Only S_2 is valid
c) Both S_1 and S_2 are valid
d) Both S_1 and S_2 are invalid

$S_1: \{\overbrace{\underbrace{(\overset{T}{\sim p} \rightarrow \overset{T}{(q \rightarrow \sim w)})}_{T} \wedge \underbrace{(\sim s \rightarrow q)}_{T} \wedge \underbrace{\sim \overset{F}{t}}_{T} \wedge \underbrace{(\overset{T}{\sim p} \vee \overset{F}{t})}_{T}}^{T}\}$

$\rightarrow (w \rightarrow s)$

Focussing only on $(q \rightarrow \sim w)$ & $(\sim s \rightarrow q)$

q	$\rightarrow \sim w$
T	T
F	T
F	F

$\sim s$	$\rightarrow q$
T	T
F	T
F	F

$$\underbrace{(\underset{\substack{\downarrow\\F}}{w} \longrightarrow \underset{\substack{\downarrow\\T/F}}{s})}_{T} \qquad \underbrace{(\underset{\substack{\downarrow\\T}}{w} \rightarrow \underset{\substack{\downarrow\\T}}{s})}_{T}$$

$S_2 : \{(q \rightarrow t) \wedge (s \rightarrow r) \wedge (\sim q \rightarrow s)\} \rightarrow (\sim t \rightarrow r)$

It is also valid

∴ option C is correct

$S_1 \rightarrow$ solution again (Different Method)

$$\underbrace{(\sim p \rightarrow (q \rightarrow \sim w))} \wedge (\sim s \rightarrow q) \wedge \underbrace{\sim t \wedge (\sim p \vee t)}$$

$\downarrow$ $\quad$ (∵ $(p \vee q) \wedge \sim q \equiv p$)

$$p \vee (q \rightarrow \sim w) \wedge (\sim s \rightarrow q) \wedge \sim p$$

$\downarrow$

$$\underbrace{(q \rightarrow \sim w) \wedge (\sim s \rightarrow q)}$$

$\downarrow$

$\equiv \sim s \rightarrow \sim w$ (Transitive)

$= w \rightarrow s$ (∵ contrapositive $p \rightarrow q \equiv \sim q \rightarrow \sim p$)

Q Which of the following is valid?

a) $\{\sim p, p \rightarrow q, q \rightarrow r\} \Rightarrow \sim r$

b) $\{p \rightarrow q, q \rightarrow r, r\} \Rightarrow p$

c) $\{p \rightarrow (q \rightarrow r), (p \wedge q)\} \Rightarrow r$

d) $\{\sim (p \wedge q), \sim p\} \Rightarrow q$

a) $\{\sim p \wedge p \rightarrow q \wedge q \rightarrow r\}$

$\{\sim p \wedge p \rightarrow r\} \not\Rightarrow \sim r$

$\sim p \wedge (\sim p \vee r) \not\Rightarrow \sim r$

b) $\{p \rightarrow r, r\} \not\Rightarrow p$

c) $[(p \wedge q) \rightarrow r] \wedge [p \wedge q] \equiv r$ (By Modus Pones)

$(\sim p \vee (\sim q \vee r)$

$\Rightarrow (\sim p \vee \sim q) \vee r$

$\Rightarrow \sim (p \wedge q) \vee r \Rightarrow (p \wedge q) \rightarrow r$

d) $(\sim p \vee \sim q) \wedge \sim p \not\equiv q$

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Q Consider the following statements

S_1: The contrapositive of

$\{(\sim r) \vee (\sim s)\} \rightarrow q$ is $\{q \vee (r \wedge s)\}$

S_2: The converse of

$\{(\sim r) \vee (\sim s)\} \rightarrow q$ is $q \rightarrow \sim(r \wedge s)$

S_3: The inverse of

$\{(\sim r) \vee (\sim s)\} \rightarrow q$ is $(r \wedge s) \rightarrow \sim q$

S_4: The negation of

$\{(\sim r) \vee (\sim s)\} \rightarrow q$ is $\sim(r \wedge s) \wedge \sim q$

Which of the following is TRUE?

a) Only S_1, S_2 and S_3
b) Only S_1, S_3 and S_4
c) Only S_2, S_3 and S_4
d) S_1, S_2, S_3, S_4

a) Contrapositive $(p \rightarrow q) \cong \sim q \rightarrow \sim p$

$\sim q \rightarrow \sim((\sim r) \vee (\sim s))$
$\sim q \rightarrow \sim(\sim(r \wedge s))$
$\sim q \rightarrow (r \wedge s)$
$q \vee (r \wedge s)$

$\because a \rightarrow b \equiv \sim a \vee b$
$\sim a \rightarrow b \equiv a \vee b$

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b) Converse of $p \rightarrow q \equiv q \rightarrow p$

c) Inverse of $p \rightarrow q \equiv \sim p \rightarrow \sim q$

$\sim\{(-r) \vee (\sim s)\} \rightarrow \sim q$
$\sim\{\sim(r \wedge s)\} \rightarrow \sim q$
$(r \wedge s) \rightarrow \sim q$

Q A binary relation $*$ is defined by the following truth table.

P	q	P * q
F	F	F
F	T	F
T	F	T
F	T	F

Then $p \rightarrow q$ is equivalent to

a) $(\sim p * q)$
b) $\sim(p * q)$
c) $\sim(p * \sim q)$
d) $(p * \sim q)$

p	q	p * q	p → q
F	F	F	T
F	T	F	T
T	F	T	F
F	T	F	T

$\therefore p \to q = \sim(p * q)$

Q The formula (Try) (Ans D)

$\{(\sim p \wedge q) \vee (p \wedge \sim q) \vee (p \wedge q)\}$

is equivalent to

a) $p \to q$ (c) $p \leftrightarrow q$

b) $p \wedge q$ (d) $p \vee q$

By Digital logic Concept

$*\, p \vee q = p + q$

$p \wedge q = p \cdot q$

$\sim p = \bar{p}$

$*\, p \vee \sim p = p + \bar{p} = 1$

$p \wedge \sim p = p \cdot \bar{p} = 0$

$$(\bar{p} \cdot q) + (p \cdot \bar{q}) + (p \cdot q)$$

$$= (\bar{p} \cdot q) + p(\bar{q} + q)$$

$$= (\bar{p} \cdot q) + p \quad (\because \bar{q} + q = 1)$$

$$= (\bar{p} + p) \cdot (q + p) \quad \text{(Distributive Law)}$$

$$= q + p$$

$$= q \vee p$$

Q The statement formula of

$\{p \wedge (\sim p \vee \sim q) \wedge (\sim p \vee q \vee r) \wedge \sim r\}$ is

a) a tautology
b) a contingency
c) not satisfiable ≡ contradiction
d) none of these

By using Quine's method

Let P = True

$$\underline{T \wedge (F \vee \sim q)} \wedge (F \vee q \vee r) \wedge \sim r$$

$$= \sim q \wedge (q \vee r) \wedge \sim r$$

$$= \sim(q \vee r) \wedge \cdot (q \vee r)$$

= false

Let P = False

Apply same process

Again
= False

$\therefore \left.\begin{array}{l} P = T \\ P = F \end{array}\right\}$ w.f.f False

∴ Contradiction

∴ Not Satisfiable

3/5/21

Q Consider the following compound proposition:-

$S_1 = ((P\to Q)\to P)\to Q$

$S_2 = P\to(Q\to(P\to Q))$

Which of the following is TRUE?

a) $S_1 \Rightarrow S_2$
b) $S_2 \Rightarrow S_1$
c) $S_1 \Leftrightarrow S_2$
d) None of these

$\Rightarrow$ L.H.S (S_1)

$= ((P\to Q)\to P)\to Q$

$= \sim(\sim(\sim P\vee Q)\vee P)\vee Q$

$= [(\sim P\vee Q)\wedge\sim P]\vee Q$

$= [(\sim P\vee Q)\vee Q]\wedge[\sim P\vee Q]$

$= (\sim P\vee Q)\wedge(\sim P\vee Q)$

$= \sim P\vee Q$

$= P\to Q$

$$\{\underbrace{((P\to Q)\to P)\to Q}_{\downarrow}\}\Rightarrow\overbrace{\{\underset{T/F}{P}\to(\overset{T/F}{Q}\to\underbrace{(P\to Q)}_{T})\}}^{T}$$

$$\underbrace{P\to Q}_{T}\qquad T\qquad T$$

$\therefore\ S_1 \Rightarrow S_2$

Q Consider the following arguments

S_1: $\{(a\vee b)\to c,\ c\to(d\wedge e)\}\to(a\to d)$

S_2: $\{(p\to q),\ \sim(p\wedge q)\}\to\sim p$

Which of the following is TRUE?

a) S_1 is valid and S_2 is not valid
b) S_2 is valid and S_1 is not valid.
c) Both S_1 and S_2 are valid
d) Neither S_1 ~~or~~ nor S_2 is valid.

$$S_1\ \{\underbrace{(\overset{T}{a}\vee\overset{T/F}{b})}_{T}\to c\wedge c\to\underbrace{(\overset{F}{d}\wedge e)}_{F}\}\to\underbrace{(\overset{T}{a}\to\overset{F}{d})}_{F}$$

$$\{\underbrace{\underbrace{\sim\underbrace{(\overset{T}{a}\vee b)}_{T}}_{F}\vee c}_{c}\wedge(\underbrace{\sim c\vee\underbrace{(\overset{F}{d}\wedge e)}_{F}}_{\downarrow})\}\to(a\to d)$$

$$\underbrace{c\ \wedge\ \sim c}_{F}$$

$\therefore$ It is not valid

S_2: $\{(p \to q) \land \sim(p \land q)\} \to \sim p$

$$\{\underbrace{(\overset{F}{\sim p} \lor q)}_{q} \land \sim\underbrace{(\overset{T}{p} \land q)}_{\sim q}\} \to \underbrace{\sim \overset{T}{p}}_{F}$$

$$\underbrace{q \land \sim q}_{F}$$

∴ It is also valid

Option C is correct

Q The statement formula $\{P \lor (P \leftrightarrow Q) \lor Q\}$ is equivalent to

a) P b) Q

c) a tautology d) $(P \land Q)$

Quine's Method

i) Let $P = T$;

$$\underbrace{\{\underbrace{P}_{T} \lor (P \leftrightarrow Q) \lor Q\}}_{T}$$

when $P = T$; w.f.f is T

ii) Let $P = F$ I^{st} Case

$$\underbrace{\{\underset{F}{P} \lor (\underset{F}{P} \leftrightarrow \underset{T}{Q}) \lor \underset{T}{Q}\}}_{T}$$

2nd case

$$\underbrace{\{\underset{F}{P} \lor \underbrace{(\underset{F}{P} \leftrightarrow \underset{F}{Q})}_{T} \lor \underset{F}{Q}\}}_{T}$$

∴ For all case it is TRUE

∴ It is a Tautology.

Conditional Proof Rule (C.P. Rule)

Given $\{P_1 \land P_2 \land P_3 \land \ldots \land P_n\} \Rightarrow (Q \to R)$ —ⓘ

then $\{\{P_1 \land P_2 \land P_3 \ldots \land P_n\} \land Q\} \Rightarrow R$ —ⓘⓘ

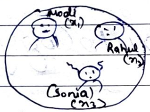
C.P Rule says that if eq(ii) is valid then eq(ii) is also valid

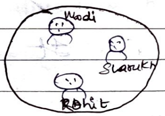
C.P Rule only applicable when implication sign is there $(Q \to R)$

First Order Logic

$P(x)$: $\underbrace{x}_{\text{Subject}}$ $\underbrace{\text{is a politician}}_{\text{Predicate(P)}}$

$\forall(x)\ P(x)$	$\exists x\ P(x)$
$\forall x\ P(x)$ ~~m~~ is TRUE means	$\exists x\ P(x)$ means
$\underbrace{\underbrace{P(x_1)}_{T} \land \underbrace{P(x_2)}_{T} \land \underbrace{P(x_3)}_{T}}_{T}$	$\underbrace{\underbrace{P(x_1)}_{T} \lor \underbrace{P(x_2)}_{F} \lor \underbrace{P(x_3)}_{F}}_{T}$



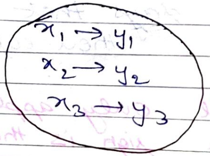


$L(x,y)$: 'x' likes 'y' (or) 'y' is liked by 'x'

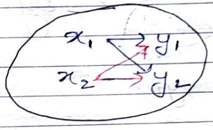
not fixed y

1) $\forall x\ \exists y\ L(x,y)$

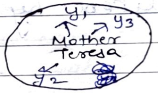
Everyone like someone



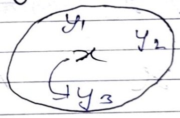
2) $\forall x\ \forall y\ L(x,y)$: Every 'x' likes every 'y'



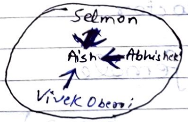
3) $\underbrace{\exists x}_{\text{fixed}}\ \forall y\ L(x,y)$: Some 'x' likes everyone ('y')



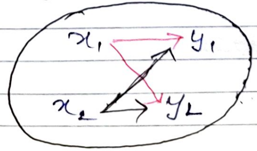
4) $\exists x\ \exists y\ L(x,y)$: Someone likes someone



5) $\exists y\ \forall x\ L(x,y)$: There is some fixed 'y' who is liked by every 'x'.

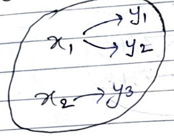


6) $\forall y\ \forall x\ L(x,y)$: Every 'y' liked by every 'x'



(for every 'y' we can find a suitable 'x' value which is need not be fixed)

7) $\forall y\ \exists x\ L(x,y)$: Every 'y' liked by some 'x'



8) $\exists y\ \exists x\ L(x,y)$

Someone liked by someone

1:28

* $\forall x \quad \longrightarrow$

$\exists x \quad \wedge$

* All doctors are female

$D(x)$: x is a doctor.

$F(x)$: x is a female

$\forall x\,(D(x) \longrightarrow F(x))$

* Some books are good

$B(x)$: x is a book

$G(x)$: x is good

$\exists x\,(B(x) \wedge G(x))$

Q What is the correct translation of the following statement into mathematical logic?

"Some real numbers are rational"

a) $\exists x\,(\text{real}(x) \vee \text{rational}(x))$

b) $\forall x\,(\text{real}(x) \rightarrow \text{rational}(x))$

c) $\exists x\,(\text{real}(x) \wedge \text{rational}(x))$

d) $\exists x\,(\text{rational}(x) \rightarrow \text{real}(x))$

Some $\exists x \quad \wedge$

$\exists x\,(\text{real}(x) \wedge \text{rational}(x))$

$\therefore$ option c correct

GATE

Q which one of the following is the most appropriate logical formula to represent the statement:

"Gold and silver ornaments are precious"
The following notations are used:

G(x): x is a gold ornament
S(x): x is a silver ornament
P(x): x is precious

a) $\forall x(P(x) \rightarrow (G(x) \wedge S(x)))$
b) $\forall x((G(x) \wedge S(x)) \rightarrow P(x))$
c) $\exists x((G(x) \wedge S(x)) \rightarrow P(x))$
d) $\forall x((G(x) \vee S(x)) \rightarrow P(x))$

$\forall x \quad \rightarrow$
$\exists x \quad \wedge$

* ∴ C is wrong (c option)

* Precious are gold & silver according to option A which is wrong
"Gold & silver are precious"

~~options~~
b) option
$\forall x$ which are both gold and silver are precious
but x can't be gold and silver at same time
∴ it is also wrong

d) option
$\forall x$ which are gold or silver are precious
∴ d option is correct.

Q Let Universe of Discourse be set of all integer numbers.

S_1: $(x^2+1) < 0$
S_2: x is odd

Which of the above statements are predicates?
a) Only S_1
b) Only S_2
c) Both S_1 and S_2
d) Neither S_1 nor S_2

Predicate is nothing but we have to replace ~~that~~ variable by every instance of domain to form a proposition

$2+3=5$ (TRUE) $\qquad$ $2+3=8$ (FALSE)

$S_1 : (x^2+1) < 0$

$\forall x \in \mathbb{Z}$, S_1 (False)
we are able to decide whether the truth value is TRUE or FALSE.

S_2: x is odd

for given value of 'x' in set of integers we can decide whether it is odd (or) even if it is odd then truth value is TRUE.

Q let $P(x,y)$ be a predicate defined as
$P(x,y) : (x \vee y) \rightarrow z$
The negation of $\forall x \exists y \, P(x,y)$ is

a) $\exists x \forall y ((x \vee y) \wedge z)$
b) $\exists x \forall y ((x \vee y) \wedge \sim z)$
c) $\exists x \forall y (\sim(x \vee y) \vee z)$
d) $\forall x \exists y ((x \vee y) \wedge \sim z)$

$\sim \forall x \equiv \exists x$;
$\sim \exists x \equiv \forall x$;

$P(x,y) : (x \vee y) \rightarrow z$
$= \sim [\forall x \exists y \, P(x,y)]$

$= \sim [\forall x \exists y [(x \vee y) \rightarrow z]]$
$= \sim [\forall x \exists y [\sim(x \vee y) \vee z]]$
$= \exists x \forall y [(x \vee y) \wedge \sim z]$

$\therefore$ Option B

Q Let D_x and D_y denote the domains of x and y, respectively.
Consider the predicate formula ϕ:

$\phi : \forall x \exists y [x + y = 17]$
Consider the following statements

1) $D_x = D_y =$ the set of integers
2) $D_x = D_y =$ the set of positive integer
3) $D_x =$ the set of integers $D_y =$ set of the integer
4) $D_x =$ the set of positive integer
$D_y =$ the set of integers.

In which of the above cases, the quantified predicate ϕ has truth value true?

a) 1 and 2   b) 2 and 3
c) 1 and 3   d) 1 and 4

$\forall x \exists y \; [x+y=17]$

$y = 17 - x$

i.e for every x we can find some y

x	y
5	12
3	14
0	17
-5	22
22	-5

∴ 1 and 4 has ~~TRU~~ TRUTH Value as TRUE.

Q Consider the following mathematical statements in number theory:

"For every integer n bigger than 1, there is a prime strictly between n and 2n"

If the universe of discourse is set of all integers and $p(x)$ denotes "x is a prime number", then which of the following first order logic ~~sentences~~ sentences correctly represents the above statements?

a) $\forall n \, [(n>1) \rightarrow \exists x \, \{p(x) \wedge (n<x<2n)\}]$

b) $\forall n \, [(n>1) \wedge \exists x \, \{p(x) \rightarrow (n<x<2n)\}]$

c) $\exists n \, [(n>1) \rightarrow \forall x \, \{p(x) \wedge (n<x<2n)\}]$

d) $\exists n \, [(n>1) \wedge \forall x \, \{p(x) \wedge (n<x<2n)\}]$

$\forall x$ follow connectivity →

$\exists x$ follow connectivity $\wedge$

B & C option eliminate

Since it is written for every integer so $\forall n$ should come

∴ D option also eliminate

∴ option A correct

Q Same as previous question But here negation of statement is asked.

a) $\exists n \, [(n>1) \wedge \forall x \{p(x) \rightarrow ((x \leq n) \vee (x \geq 2n))\}]$

b) $\forall n \, [(n>1) \wedge \exists x \, \{p(x) \rightarrow ((x \leq n) \vee (x \geq 2n))\}]$

c) $\exists n \, [(n>1) \rightarrow \forall x \, \{p(x) \wedge ((x \leq n) \vee (x \geq 2n))\}]$

d) $\forall n \, [(n>1) \rightarrow \exists x \, \{p(x) \wedge ((x \leq n) \vee (x \geq 2n))\}]$

$= \forall n[(n>1) \rightarrow \exists x(P(x) \wedge (n<x \wedge x<2n)]$

$= \forall n[\sim(n>1) \vee \exists x(P(x) \wedge (n<x \wedge x<2n)]$

$\Rightarrow$ Apply '$\sim$'

$\Rightarrow \exists n[(n>1) \wedge \forall x(\sim P(x) \vee (x \leq n \vee x \geq 2n])$

$\Rightarrow \exists n[(n>1) \wedge \forall x(P(x) \rightarrow (x \leq n \vee x \geq 2n])$

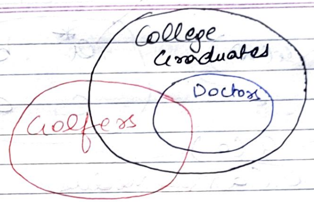
Q Consider the following arguments.

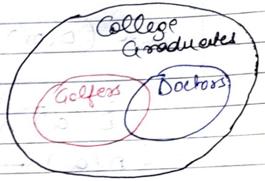
I) All doctors are college graduates
Some doctors are not golfers
Hence, some golfers are not college graduates.

II No mothers are males.
Some males are politicians
Hence, some politicians are not mothers

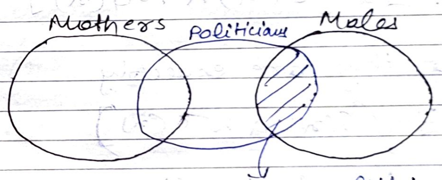
Which of the following is true?

a) Both argument are valid
b) Both argument are Invalid
c) Only argument I is valid
d) Only argument II is valid.





As two cases are possible therefore it is invalid
(we cannot Decide)



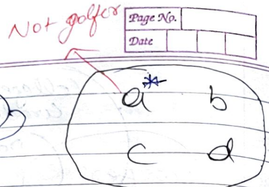
Some politician are not mothers

∴ Option D is correct

Another Method

D(x) : x is a Doctor
C(x) : x is a college graduate
G(x) : x is a golfer

1.) $\forall x\ (D(x) \rightarrow C(x))$
2) $\exists x\ (D(x) \wedge \sim G(x))$



$\exists x\,(D(x) \land \sim G(x))$

$D(a) \land \sim G(a)$

$D(a) \to C(a)$ → Modus Pones

$C(a) \land \sim G(a)$

$\therefore \exists x\,[C(x) \land \sim G(x)]$

But they are expecting

$\exists x\,(G(x) \land \sim C(x))$

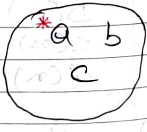
∴ this is invalid

for IInd Statement

M(x) : x is a Mother
N(x) : x is a Male
P(x) : x is a Politician

$\forall x\,[M(x) \to \sim N(x)]$

$\exists x\,[N(x) \land P(x)]$



$(N(a) \land P(a))$ Disjunctive syllogism

$M(a) \to \sim N(a) \equiv \sim M(a) \lor \sim N(a)$

$\sim M(a) \land P(a)$

$\therefore \exists x\,[P(x) \land \sim M(x)]$

which is expected argument

∴ It is valid

Option D is correct.

Q Let P and Q be two predicates. Consider the following statements

$S_1: \exists x\,[P(x) \lor Q(x)] \Leftrightarrow (\exists x\, P(x) \lor \exists x\, Q(x))$

$S_2: \forall x\,[P(x) \land Q(x)] \Leftrightarrow (\forall x\, P(x) \land \forall x\, Q(x))$

Which of the following is True?

a) Only S_1
b) Only S_2
c) Both S_1 and S_2
d) Neither S_1 nor S_2

Standard Result

$\exists x$ distributive over ($\lor$) (or)

$\forall x$ distributive over ($\land$) (and)

∴ Both statements are TRUE

∴ Option C is correct

Q What is the first order predicate calculus statement equivalent to the following?

'Every Teacher is liked by some student'

✗ a) $\forall(x)[\text{teacher}(x) \rightarrow \exists(y)[\text{student}(y) \rightarrow \text{likes}(y,x)]]$

b) $\forall(x)[\text{teacher}(x) \rightarrow \exists(y)[\text{student}(y) \wedge \text{likes}(y,x)]]$

✗ c) $\exists(y)\forall(x)[\text{teacher}(x) \rightarrow [\text{student}(y) \wedge \text{likes}(y,x)]]$

✗ d) $\forall(x)[\text{teacher}(x) \wedge \exists(y)[\text{student}(y) \rightarrow \text{likes}(y,x)]]$

likes(y, x): y likes x (or)

x is liked by 'y'

Since some students are not fixed

$\therefore \exists(y)\forall(x)[\_\_ \;\; \_\_]$

(i.e c option is not correct)

$\therefore$ option B is correct

~~Test~~

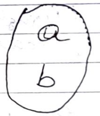
Q Let $P(x)$ and $Q(x)$ be arbitrary predicates, which of the following is always TRUE?

a) $(\forall x(P(x) \vee Q(x))) \rightarrow ((\forall x P(x)) \vee (\forall x Q(x)))$

b) $(\forall x(P(x) \rightarrow Q(x))) \Rightarrow ((\forall x P(x)) \rightarrow (\forall x Q(x)))$

c) $((\forall x(P(x)) \rightarrow (\forall x(Q(x))) \Rightarrow (\forall x(P(x) \Rightarrow Q(x)))$

d) $(\forall x(P(x))) \leftrightarrow (\forall x(Q(x)))) \Rightarrow (\forall x(P(x) \Leftrightarrow Q(x)))$



$P(x)$: x is politician

$Q(x)$: x is actor

Assume that L.H.S is TRUE on the above domain

i.e $\forall x(P(x) \rightarrow Q(x))$ is T

i.e $[P(a) \rightarrow Q(a)] \wedge [P(b) \rightarrow Q(b)]$

Now R.H.S

$(\forall x P(x)) \rightarrow \forall x Q(x))$

$[P(a) \wedge P(b)] \rightarrow [Q(a) \wedge Q(b)]$

Option B is standard result correct for all cases

$\therefore$ option B correct

Another method of proving option B

C.P Rule

i) $\{P_1 \cancel{Q}, P_2, P_3, \ldots, P_n\} \to (Q \to R)$

ii) $\{\{P_1 \wedge P_2 \wedge P_3 \ldots \wedge P_n\} \wedge Q\} \to R$

$\forall x (P(x) \to Q(x)) \Rightarrow \{\forall x\, P(x) \to \forall x\, Q(x)\}$

$\{\forall x (P(x) \to Q(x)) \wedge \forall x\, P(x)\} \Rightarrow \forall x\, Q(x)$

$\therefore \forall x\, Q(x)$ [Using Modus Pones]

Q which one of the first order predicate calculus statements given below correctly expresses the following English statement?

Tigers and lions attack if they are hungry or threatened.

a) $\forall x[(tiger(x) \wedge lion(x))$

$\to \{(hungry(x) \vee threatened(x)) \to attacks(x)\}]$

b) $\forall x[(tiger(x) \wedge lion(x))$

$\to \{(hungry(x) \vee threatened(x)) \to attacks(x)\}]$

c) $\forall x[(tiger(x) \vee lion(x))$

$\to \{attacks(x) \to (hungry(x) \vee threatened(x))\}]$

d) $\forall x[(tiger(x) \vee lion(x))$

$\to \{(hungry(x) \vee threatened(x)) \to attacks(x)\}]$

$\forall(x)[(tiger(x) \wedge lion(x))]$

x is same in both case ~~and~~ but tiger and lion cannot be same

∴ option a & b is eliminated

c option says
tiger or lion ~~attacks are~~ attacks ~~as~~ ~~they are~~ means they are hungry or threatned (they can attack for fun purpose also :))

∴ C option is also wrong.

D option is correct

Q Consider the following first order logic formula in which R is a binary relation symbol.

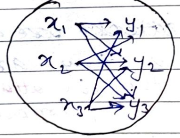
$\forall x \forall y (R(x,y) \Rightarrow R(y,x))$.

The formula is

a) satisfiable and valid
b) satisfiable and so is its negation
c) unsatisfiable but its negation is valid
d) satisfiable but its negation is unsatisfiable.

$R(x,y)$: x likes y

$R(y,x)$: y likes x



x likes y but we can't guarantee that y likes x

∴ It can be True or False

∴ It is not valid but it is satisfiable

* ~~If any pred~~

* Whenever a predicate is satisfiable it's negation is also satisfiable.

Q Let Graph(x) be a predicate which denotes that x is a graph. Let Connected(x) be a predicate which denotes that x is connected. Which of the following first order logic sentences DOES NOT represent the statement:
"Not every graph is connected"?

a) $\sim\forall x(\text{Graph}(x) \Rightarrow \text{connected}(x))$

b) $\exists x(\text{Graph}(x) \wedge \sim\text{connected}(x))$

c) $\sim\forall x(\sim\text{Graph}(x) \vee \text{connected}(x))$

d) $\forall x(\text{Graph}(x) \rightarrow \sim\text{Connected}(x))$

Not All/Every graph are connected

$\sim \forall x [G(x) \longrightarrow C(x)]$

∴ D option does not represent the statement

Q Let fsa and pda be two predicates such that fsa(x) means x is a finite state ~~au~~ automation and pda(y) means that y is a pushdown automation. Let equivalent be another predicate such that equivalent (a,b) means a and b are equivalent. which of the following

"Each finite state automation has an equivalent pushdown automation".

a) $(\forall x \, fsa(x)) \Rightarrow (\exists y \, pda(y) \wedge equivalent(x,y))$

b) $\sim \forall y \, (\exists x \, fsa(x) \Rightarrow pda(y) \wedge equivalent(x,y))$

c) $\forall x \, \exists y \, (fsa(x) \wedge pda(y) \wedge equivalent(x,y))$

d) $\forall x \, \exists y \, (fsa(y) \wedge pda(x) \wedge equivalent(x,y))$

Option A is correct

Q Consider the following well formed formulae:

I. $\sim \forall x \, (P(x))$

II. $\sim \exists x \, (P(x))$

III. $\sim \exists x \, (\sim P(x))$

IV. $\exists x \, (\sim P(x))$

which of the above are equivalent?

a) I and II
b) I and IV
c) II and III
d) II and IV

I. $\sim \forall x \, (P(x))$

$\exists x \, (\sim P(x)) = \text{IV}$

∴ I & IV are equivalent

B option is correct

Q Suppose the predicate $F(x, y, t)$ is used to represent the statement that person x can fool person y at time t. which one of the statements below expresses best the meaning of the formula

$\forall x \, \exists y \, \exists t \, (\sim F(x, y, t))$?

a) Everyone can fool some person at some time.
b) No one can fool everyone all the time
c) Everyone cannot fool some person all the time.
d) No one can fool some person at some time.

$\forall x\, \exists y\, \exists t (\sim F(x, y, t))$

$\sim [\exists x\, \forall y\, \forall t (F(x, y, t))]$

$\downarrow$

Not (Someone can fool everyone at all time)

= No person can fool everyone at all time

$\therefore$ option B is correct

Q which one of the following option is correct given three positive integers x, y and z and a predicate.

$P(x) : \sim(x=1) \wedge \forall y \{ \exists z (x = y * z) \Rightarrow (y = x) \vee (y = 1) \}$

a) P(x) being True means x is a prime number.
b) P(x) being True means that x is a number other than 1.
c) P(x) is always true irrespective of the value of x.
d) P(x) being true means that x has exactly two factors other than 1 and x.

~~It~~ Given ~~a for~~ expression is the definition of prime number

$\therefore$ option A is correct.

Q what is the logical translation of the following statement?

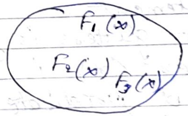
"None of my friends are perfect"

a) $\exists x (F(x) \wedge \sim P(x))$

b) $\exists x (\sim F(x) \wedge P(x))$

c) $\exists x (\sim F(x) \wedge \sim P(x))$

d) $\sim \exists x (F(x) \wedge P(x))$



x → Not perfect
F_n → friend

$\forall x (F(x) \longrightarrow \sim P(x))$

$\forall x (\sim F(x) \vee \sim P(x))$

$\forall x (\sim (F(x) \wedge P(x))$

$\sim [\exists x (F(x) \wedge P(x)]$

Option D is correct

Q which one of the following is NOT logically equivalent to
$\sim \exists x(\forall y(\alpha) \wedge (\forall z(\beta)))$?

a) $\forall x(\exists z(\sim\beta) \rightarrow \forall y(\alpha))$

b) $\forall x(\forall z(\beta) \longrightarrow \exists y(\sim\alpha))$

c) $\forall x(\forall y(\alpha) \longrightarrow \exists z(\sim\beta))$

d) $\forall x(\exists y(\sim\alpha) \longrightarrow \exists z(\sim\beta))$

A A option is correct

Q Consider the statement
"Not all that glitters is gold"
Predicate glitters(x) is true if x glitters and predicate gold(x) is true if x is gold.

which one of the following logical formulae represents the above statement?

1) $\forall x$: glitters(x) $\Rightarrow \sim$gold(x)
2) $\forall x$: gold(x) $\Rightarrow$ glitters(x).
3) $\exists x$: gold(x) $\wedge \sim$ glitters(x)
4) $\exists x$: glitters(x) $\wedge \sim$gold(x)

* Not all graphs are connected (Some graphs are connected)
* All graphs are not connected
Both statements are different

Not all that glitter is gold
(i.e some glitter is gold)
$\therefore$ $\exists x$: glitters(x) $\wedge \sim$ gold(x)
$\therefore$ option D is correct

o———o———o———o———

1) ~~level~~ Target :- 75

2) Read Topic wise

3) Divide Topics in High priority & low priority based on previous years GATE paper

4) level 1 → Read Class notes

level 2 → Solve 1987-2021 prev. year GATE Ques.

level 3 → Solve Workbook Question

level 4 → ~~[illegible]~~ Online Test Series (in LAST)